Mathematics B-day 2025

Jumping bucks

Universiteit Utrecht

Matematický B-deň

Wiskunde voor teams

INTRODUCTION

ABOUT THE ASSIGNMENT

Some puzzles or games can be explained in just two sentences, yet they conceal entire worlds. Today, you'll explore something like this: a field of squares is filled with tokens (bucks) that can jump over each other. How far can the bucks penetrate the empty space? We hope you won't shy away from doing your own creative jumps, and will ultimately arrive at sparkling insights accompanied by sound reasoning!

STRUCTURE OF THE DAY

This Math B-day assignment consists of introductory exercises and supplementary, in-depth exercises. Unlike regular math lessons, you certainly don't have to solve all the problems. There are problems ranging from easy to more difficult. It's normal that you won't complete everything, but at least show in your report what you've attempted and how far you've come. If you've spent enough time on the introductory problems, choose one or more of the final investigations to delve deeper into a topic. Gaining insights into these final problems will really set your team apart!

WORKING IN TEAMS

What's special about the Math B-day is that you'll be doing math as a team. It might be helpful to create a schedule and divide the tasks. Let each team member focus on what they're good at. Give everyone the opportunity to contribute ideas and solutions. You can work individually on different or the same problems simultaneously, then come back together to discuss and evaluate. For some problems, it's helpful to study a few different examples. This allows for a more even distribution of the work.

SUPPLIES

Today you'll need the following: a pen, plenty of (scratch) paper, this assignment, and a computer or laptop to write your report. Using information from the internet is strictly prohibited, with the exception of the links to the game boards we share in this document.

WHAT DO YOU DELIVER?

You'll be working on a digital report throughout the day. Don't start too late; you'll submit it promptly at 4:00 PM. In this report, you'll describe your results and reasoning. Tell your own clear and compelling story. We appreciate well-written, clear, precise, complete, carefully worded, and certainly original, creative, and lyrical reports.

Tips:

- It can be useful to start writing parts of the final report in the morning.
- *Be understandable* : Make sure the text is readable for someone who didn't participate in the Mathematics B-day (but does have a sufficient command of mathematics), *even if they haven't read the assignment*. Preferably, don't copy the assignments verbatim into the report. Instead, create a continuous, creative narrative .
- Exploration and reasoning are the heart of the Mathematics B-day. When providing substantiation, explanations, or clarifications, try to do so *using mathematical arguments as much as possible* . The more precise and detailed your reasoning, the better. If you still have

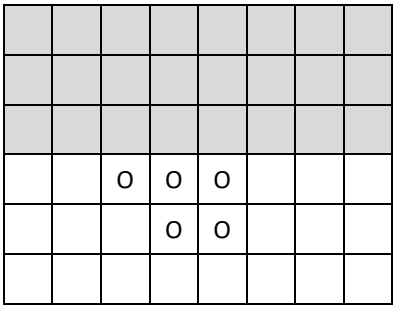
doubts about something, you can also indicate this in the report: "We suspect or hypothesize that...". That can also be very valuable.

- Use *figures* to illustrate your ideas. For example, use copies of drawings you've made (screenshots or photos of figures on paper).

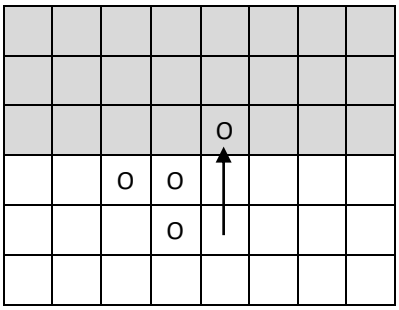
Both the mathematical content of the report and the way it is written count towards the assessment!

INTRODUCTORY PROBLEMS

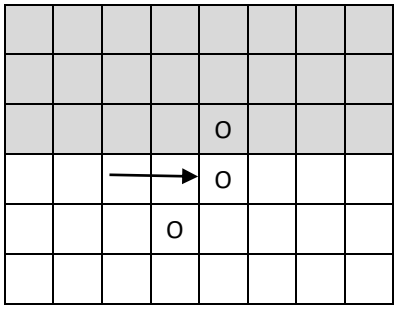
Today you'll explore variations of the following one-player game. The first variation is played on a board with square fields. A field can be filled with a round token (buck) or not. This results in a *layout* of bucks, as shown below:



The only move allowed is the *buck jump*: a buck jumping over a buck horizontally or vertically adjacent to it. The buck must land on an empty space. So, to the left, right, up, or down; jumping diagonally is not allowed. The buck that is jumped over then disappears. For example: the buck in the bottom right jumps over the buck above it and lands two spaces above its starting space. There is now one fewer buck in the game.



A next move could be: the leftmost buck jumps over the buck to the right of it and lands two spaces to the right.



At the beginning of the game, a certain number of bucks may be placed in the white squares: the *starting position* . After that, only buck jumps are allowed; no more bucks are added. The goal is to get as far as possible into the gray area. You can use the online game board here:

https://www.bden.fpvai.ukf.sk/squares_game.php?language=en

If necessary, zoom out to see the whole board and the reset-button.

ASSIGNMENT 1

See the starting grid below.

a. Check that this gets you to the second gray row.

3							
2							
1							
		O	O	O			
				O			

b. Argue that with a starting position of only three bucks, you will not get further than the first gray row.

The next question is, of course, whether you can reach the third row, and if so, with how many bucks and in what starting position. You could, of course, just try things out first, but there's a smarter approach. You now know that you can jump to row 2 with the starting position above. If you can get it to look like the figure below after a few bucks, then you should be able to reach the third row from there.

3							
2							
1		O	O	O			
				O			

So the question now is: how do you create a starting position that, after a number of jumps, results in the result shown above?

ASSIGNMENT 2

a. Find such a starting position.[1]

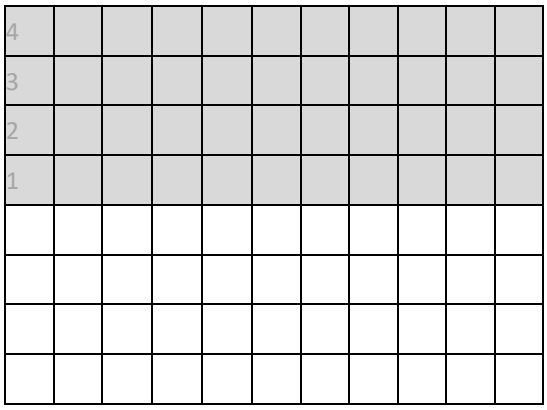
The challenge today will not only be to get as far as possible, but also to do it with **as few bucks as possible** .

b. Find a starting position that reaches the third row with the fewest possible bucks. Explain how you arrived at this. Don't try to argue yet that it really can't be done with fewer bbucks (that will come later). Note: perhaps your solution in part a was already the minimum, in which case you don't need to do anything else!

It even turns out to be possible to reach the fourth row (although with a different starting position).

[1]You can also use a few arrows and intermediate scores in the report to show which jumps are needed to reach the third row.

c. Find a starting position that reaches the fourth row with as few bucks as possible. Explain how you arrived at this. You might need a slightly larger field, as shown below. Again, don't try to argue yet that it really can't be done with less (that will come later).



The fifth row is unreachable. So we won't ask you to try to achieve it, because that's a waste of time. Instead, in the tasks below, we'll gradually develop a line of reasoning to demonstrate this. Pay close attention, because a similar line of reasoning might be useful in your own research later on the day.

Intermezzo: the Fibonacci sequence

A central role in the reasoning is reserved for the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … As you have probably already noticed, in the Fibonacci sequence each number in the sequence (except of course the first two) is the sum of its two predecessors.

You can use the Fibonacci numbers to cleverly demonstrate that you can never reach the third row with seven bucks. Let's argue that with the following starting position, you can't reach the dark gray square:

3										
2										
1										
			O	O	O	O				
				O	O					
					O					

We can calculate this using a so-called *Fibonacci triangle* (*F-triangle*). An example of such an F-triangle is shown in the figure below. In such an F-triangle, each column contains a Fibonacci row, but each step away from the middle column that row begins one square higher. You always place the F-triangle so that the starting position is completely within the triangle and also so that the bucks don't jump out with the intended jumps. Part of the F-triangle can fall of the board in the corners.

1	2	3	5	8	13	21	13	8	5	3	2	1
1	1	2	3	5	8	13	8	5	3	2	1	1
	1	1	2	3	5	8	5	3	2	1	1	
		1	1	(2)	(3)	(5)	(3)	2	1	1		
			1	1	(2)	(3)	2	1	1			
				1	1	(2)	1	1				
					1	1	1					
						1						

The *weight* of a set of bucks relative to a chosen F-triangle is the sum of the numbers in the squares where a buck lies; in this example:

$$2 + 2 + 3 + 2 + 3 + 5 + 3 = 20.$$

Compare the weight of position before and after a jump. What do you notice?

ASSIGNMENT 3

Explain that with a jump the weight of a postion can only be reduced or remain the same (not just for the position in the example, but for all possible position).

It immediately follows that with the starting position in the example above, the target square can never be reached. The weight of the starting position is 20, while the weight of the final position, with at least one buck on the target square, must be at least 21.

Given a starting position, multiple F-triangles are possible. For the rest, it doesn't matter which F-triangle you choose, as long as it meets the above conditions. Verify that the above reasoning also works if you move the F-triangle one square up or one square down: only the weights are different.

Let's practice this way of reasoning some more, because it will come in handy in the rest of the assignment.

With the starting position below, you can't reach the target square. You can easily see this by trying (do that), but you can also reason it out with F-triangles. We place the F-triangle as shown below. Calculate for yourself that it's impossible. Also check this if you place the triangle a square higher or lower.

2	3	5	8	13	21	34	21	13	8	5	3	1
1	2	3	5	8	13	21	13	8	5	3	2	1
1	1	2	3	5	8	13	8	5	3	2	1	1
	1	1	2	3	(5)	(8)	5	3	2	1	1	
		1	1	2	3	(5)	3	2	1	1		
			1	1	2	(3)	2	1	1			
				1	1	(2)	1	1				
					1	(1)	1					
						(1)						

It seems that you cannot reach the third row with any starting grid of seven bucks.

ASSIGNMENT 4

Give a complete reasoning why seven bucks can never reach the third row.

ASSIGNMENT 5

a. Give a complete reasoning why the fourth row can never be reached with 17 bucks.
b. Can you further refine the reasoning to 18, 19, … bucks?

As previously announced, it's impossible to reach the fifth row, no matter how many blocks you use! The essential ingredient for reasoning this is that, no matter how you arrange the Fibonacci triangle, the sum of the numbers in the white squares is always smaller than the number in the target square on the fifth row. To reason this, you need a special property of the Fibonacci sequence.

A special property of the Fibonacci sequence . Look at the numbers you get by adding the sequence:

$$1 = 1$$
$$1 + 1 = 2$$
$$1 + 1 + 2 = 4$$
$$1 + 1 + 2 + 3 = 7$$
$$1 + 1 + 2 + 3 + 5 = 12$$
$$1 + 1 + 2 + 3 + 5 + 8 = 20$$

You'll notice this: the sum of the first 4 Fibonacci numbers is equal to the 6^{th} Fibonacci number minus 1; the sum of the first 5 Fibonacci numbers is equal to the 7^{th} Fibonacci number minus 1; and so on.

ASSIGNMENT 6

Reason why the sum of the first $n$few Fibonacci numbers is equal to the $(n + 2)$-th Fibonacci number minus 1.

Even if this assignment is not successful, you may use the result in the future.

ASSIGNMENT 7

Provide a complete reasoning why you cannot reach the fifth row with any (finite) starting position.

OWN RESEARCH

We invite you to choose one (or more) of the topics below for your own research.

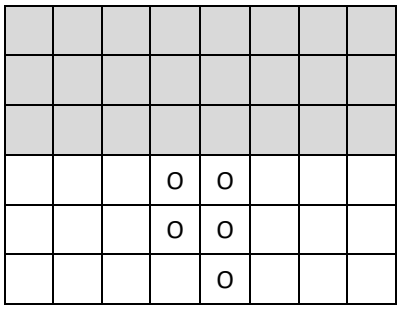
As a reminder, the research report consists of an introduction based on your findings from the basic assignments. You will then describe your approach and results for at least one of the studies you choose from the options below.

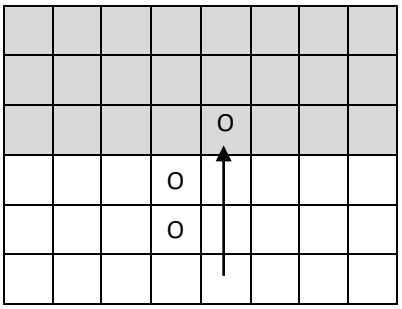
In the research you will always ask the same kind of questions as in the morning's assignments:

- With how many bucks, at a minimum, can I reach the 2nd, 3rd, 4th, etc. row? What does the corresponding starting lineup look like? Is it unique, or are there several possibilities?
- How do you prove that you can't do with less?
- Which row cannot be reached with a given number of bucks, or regardless of the number of bucks? How do you prove this?

VARIANT 1

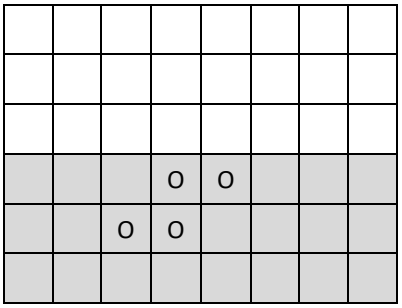
A buck jump is not across one but across two other bucks, horizontally or vertically:

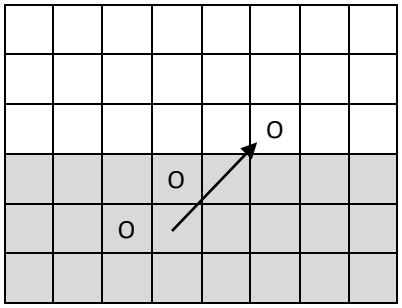




VARIANT 2

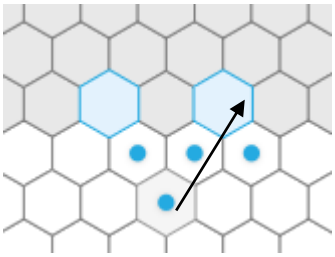
A buck only jumps diagonally (so no longer horizontally or vertically), in four directions:

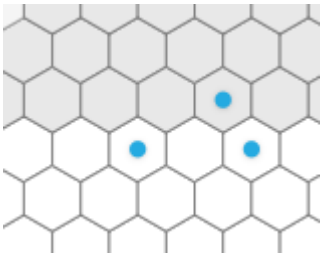




VARIANT 3

The grid is hexagonal, and a jump can go in six directions:





You can use the online game board here:

https://www.bden.fpvai.ukf.sk/hexagon_game.php?language=en

Note the underscore "_" between "hexagon" and "game". If necessary, zoom out to see the whole board and the reset-button.

FREE VARIANT

Make up your own variation of the game and research it in a similar way.

Note: Using weights to reason about whether a particular row can be reached with a certain number of bucks is very powerful, but not always the "final word." A fictional example: suppose the most conclusive proof you can provide with weights is that you can't reach the sixth row with 23 stones. Suppose you've found a starting position that allows you to reach the sixth row with 25 stones. Two things could still be wrong: it can or it can't be done with 24 bucks. If it can't be done, you'll have to devise another reasoning method, perhaps quite different from the one using weights. No one's stopping you!