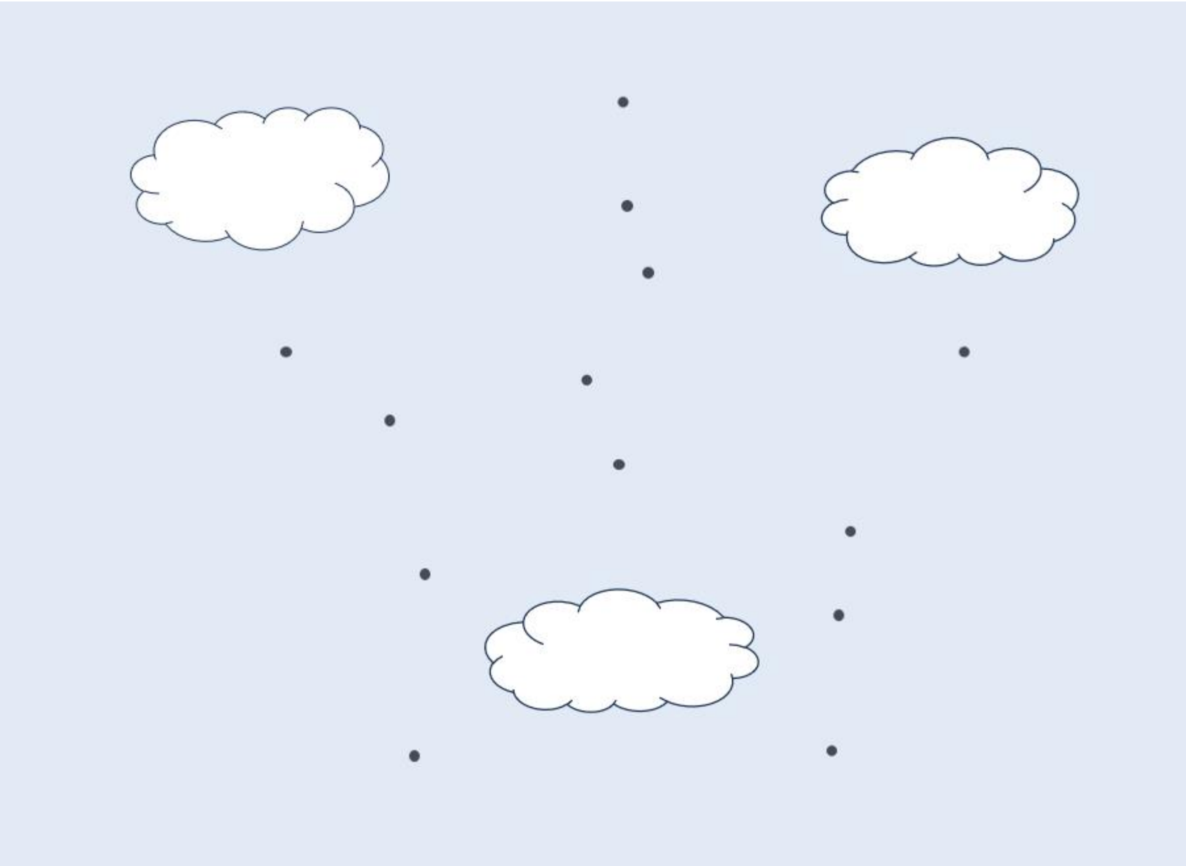
Mathematics B-day 2024



In the clouds

Universiteit Utrecht

Wiskunde voor teams

Freudenthal Institute

INTRODUCTION

ABOUT THE ASSIGNMENT

What do you see when you look into the clouds? A sheep, a dragon? Mathematicians can also stare endlessly into the clouds, point clouds that is, or a cluster of points in the plane. Today the sport is to see so-called convex polygons in them. A very easy problem to explain, but also a problem of unfathomable depths. Enough to amuse you with for a day!

STRUCTURE OF THE DAY

This Mathematics B-day assignment consists of introductory assignments and additional, in-depth assignments. Unlike normal math lessons, you certainly do not have to solve all the problems. There are problems ranging from easy to difficult. It is normal that you will not finish everything, but at least show in the report what you have tried and how far you have come. If you have spent enough time on the introductory problems, choose one or more of the final investigations to delve deeper into a topic. With insights into these final problems, your team can distinguish itself even more!

WORKING IN TEAMS

The special thing about the mathematics B-day is that you do mathematics as a team. It might be useful to make a schedule and a division of tasks. Let each team member do what they are good at. Give everyone the space to contribute with ideas and elaborations. You can work individually on different or the same problems at the same time and then come together again to discuss and evaluate. For some problems, it is useful to study multiple examples. That work can then be nicely divided.

SUPPLIES

Today you will need the following: a pen, enough (scrap) paper, this assignment, a computer or laptop to write your report. The use of information from the internet is not allowed.

WHAT DO YOU CONTRIBUTE?

You will work on a digital report during the day. Do not start too late; you will hand it in at 16:00 sharp. In this report, you will describe your results and reasoning. Tell your own, clear and convincing story. We appreciate well-written, clear, precise, complete, carefully formulated, and certainly also original, creative, and lyrical reports.

Tips:

- It may be useful to start writing parts of the final report in the morning.
- *Be understandable*: make sure that the text is readable for someone who did not participate in the Mathematics B-day (but does have sufficient mathematics), *without having read the assignment*. You do not have to literally copy the assignments into the report. Instead, make it a running, creative story.
- Explorations and reasoning are the heart of the Mathematics B-day. If you provide substantiations, explanations, or statements, try to do so *with mathematical arguments as much as possible*. The more precise your reasoning is and the more details your reasoning

provides, the better. If you still have doubts about something, you can also indicate this in the report: "We suspect that...". That can also be very valuable.

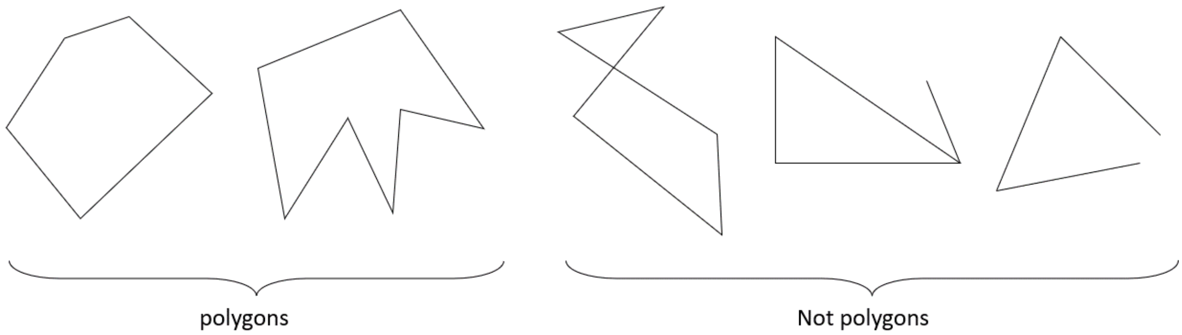
- Use *figures* to illustrate your ideas. For example, use copies of drawings you have made (screenshots or photographs of figures on paper).

Both the mathematical content of the report and the way it is written count towards the assessment!

Introductory issues

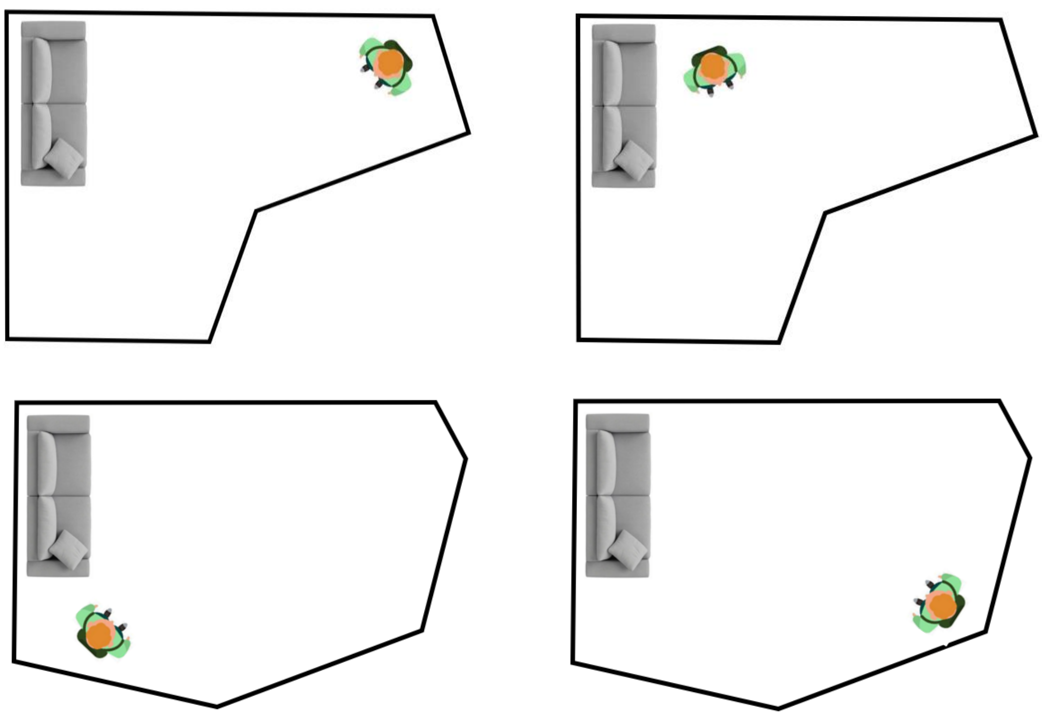
You probably already know what a polygon is: it is a finite series of line segments, where

- each line segment begins at the vertex where the previous one ends and the last line segment ends where the first one begins
- no two line segments intersect



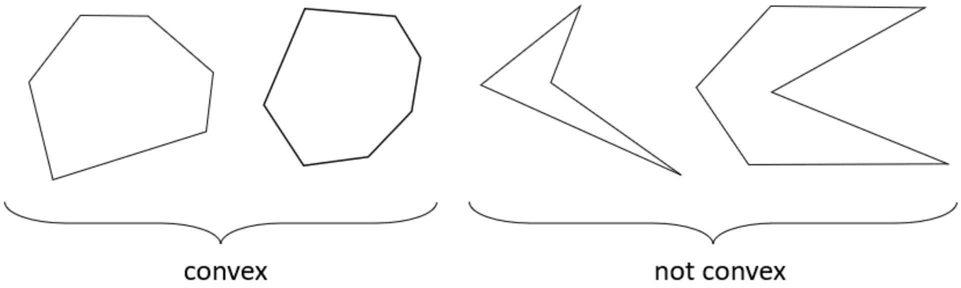
Exploration 1

Suppose a room has the shape of a polygon in top view. Under what condition(s) on that polygon can you see the entire room **from any point** in the room? Explain how you arrive at those conditions.

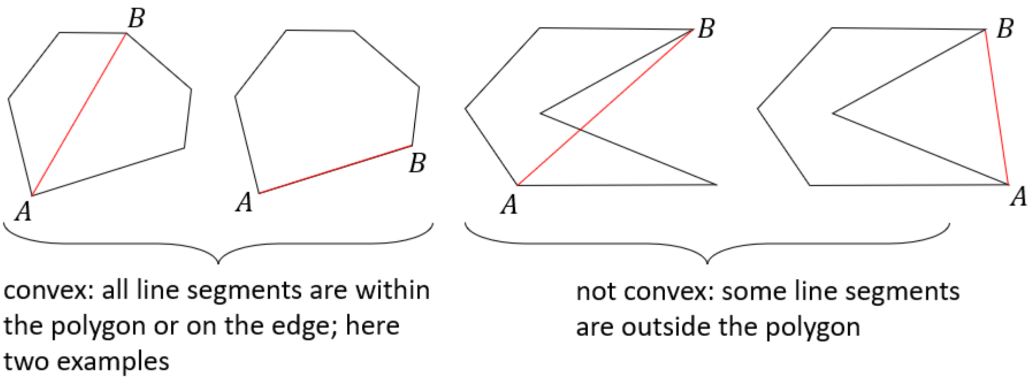


CONVEX POLYGONS

A polygon is **convex** if every interior angle is less than 180 degrees[1].



An equivalent description of convex polygons is as follows: for any pair of vertices A and, B the line segment lies AB entirely within the polygon (or at least on the edge).



Yet another characterization of convex polygons is that for each side, the polygon lies entirely on one side of that side.

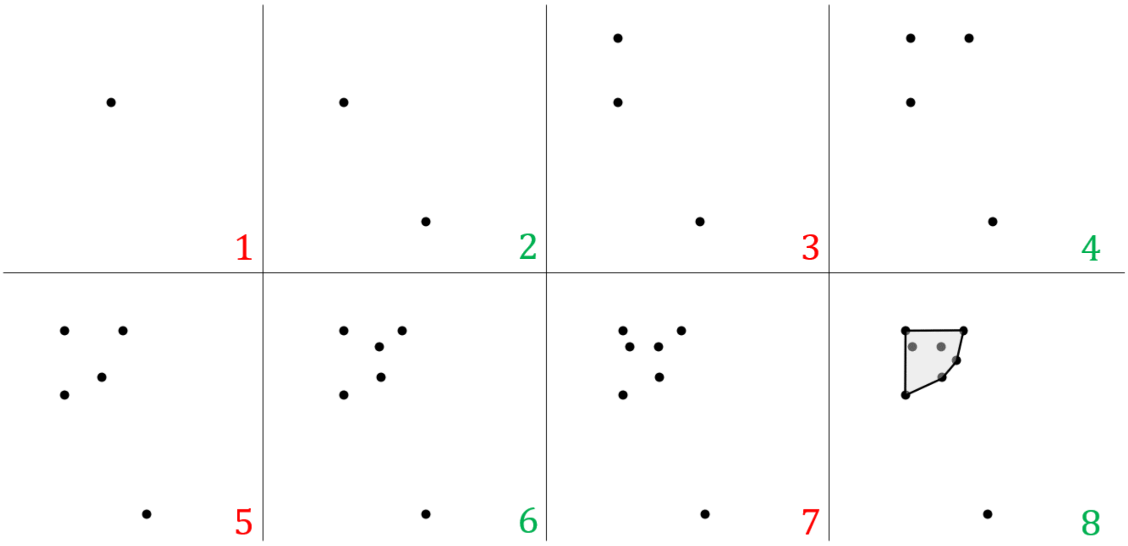
A GAME WITH CONVEX POLYGONS

We now describe a simple game for two players in which convex pentagons play a central role. The requirements are a sheet of paper and writing utensils. These are the rules:

1. The players take turns placing a point on the sheet
2. No three points may lie on a line
3. When a player makes it possible with a move to form a convex pentagon with the points, that player has lost

The next page shows a possible game progression.

[1] Normally a distinction is made between convex and strictly convex. What we call convex here is actually strictly convex. The difference is that normally convex polygons allow interior angles of 180 degrees, or three vertices on a line. These types of polygons are not discussed in this mathematics B-day, and therefore we will simply stick to the term "convex".



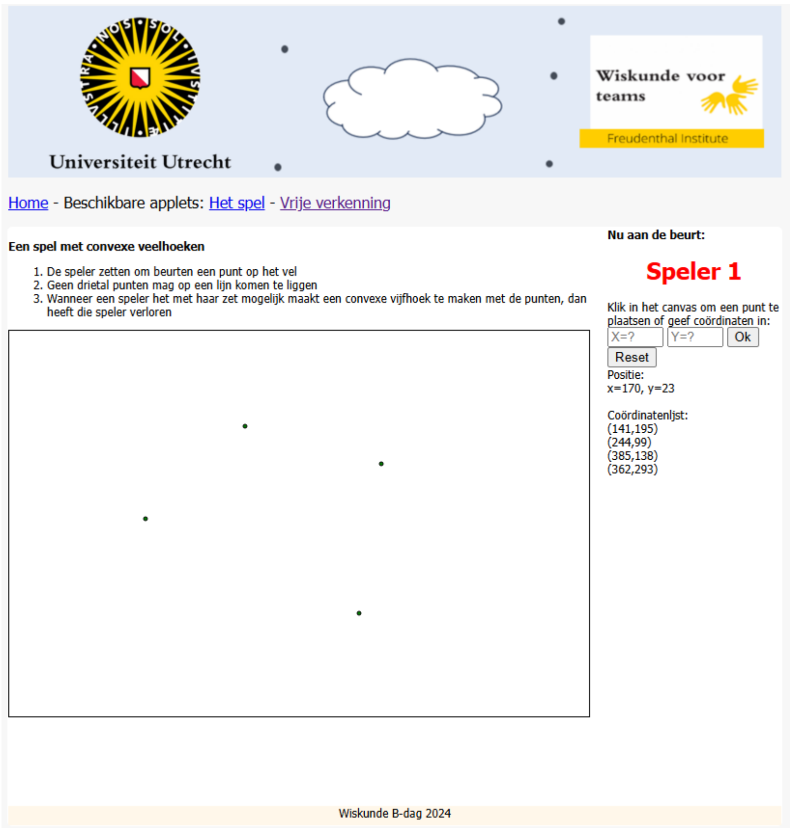
The first player wins, because the second player, by placing the last point, makes a convex pentagon possible (and even a second one).

Problem 1 (Trying it out)

Play the game! On paper or here: https://www.fisme.science.uu.nl/toepassing/29255/opdracht.html.

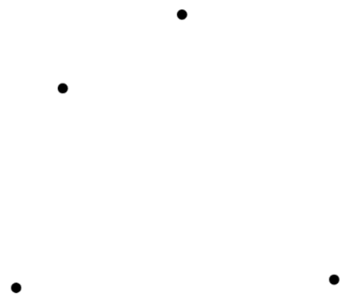
Write down each time how long the game lasts. What do you notice?

Note: Don't play too long, 5-10 minutes max! There's still plenty to do.



EXPLORATION 2

Once a first convex quadrilateral has been formed during the game, you have to be careful. There are four areas where a new point cannot end up. Draw the boundaries of these areas and shade them for the points below.



PROBLEM 2 (COOPERATIVE PLAY)

You can also cooperatively play the same game. That is, you try to place as many points as possible together before a convex pentagon is formed.

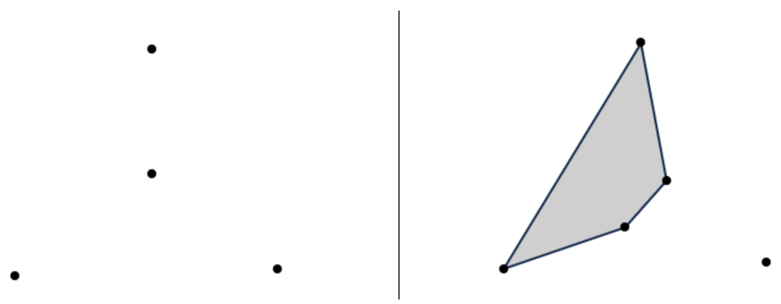
> Explore how many moves you can make the game last at most. Write down a guess (you will investigate this guess further during the rest of the day).

Note: Don't play too long, 5-10 minutes max! There's still plenty to do.

A SIMPLER VARIANT WITH CONVEX QUADRILATERALS

You can also play the game (in the cooperative variant) with convex quadrilaterals instead of pentagons. The game then becomes a bit dull. The question is: with at least how many points in the plane is a convex quadrilateral guaranteed to form? Just try it. Spoiler alert: the answer is “five”.

Just look below on the left: four is still possible. But with five a convex quadrilateral is always possible. The picture on the right is an example that it goes wrong with five points. Give a few more examples of that yourself.

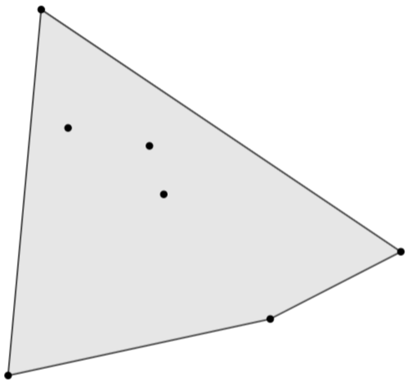


You can also play the convex quadrilateral game in the app. To do so, go to “Vrije verkenning” (meaning “free exploration”, at the top) and select “N-hoek” [4] (N-polygon).

CONVEX HULL

A collection of points in the plane, as in the game, is called a *point cloud*. We call a point cloud in which no three points lie on a line a *good point cloud*. We always work with such good point clouds today, even if we do not explicitly mention this.

Given a good point cloud, you can draw a convex polygon through the "outer" points, that is, such that all other points fall within this polygon. This is called the convex hull of the points.



If the points were nails in a board and you were to stretch a rubber band wide around all the nails, the rubber band would pull taut exactly on the convex shell.

PROBLEM 3 (REASONING ABOUT CONVEX QUADRILATERALS)

Consider the following statement:

In a good five-point cloud, there are always four points that form a convex quadrilateral.

Below we will help you give a reasoning as to why this is the case.

With a good five-point point cloud, there are three possibilities:

i. The convex hull is a pentagon

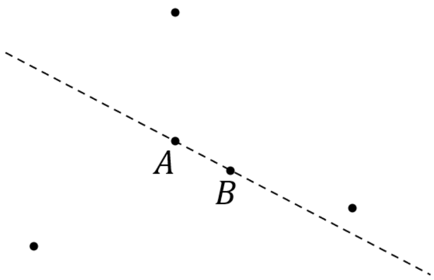
ii. The convex hull is a quadrilateral
iii. The convex hull is a triangle

a. Give an example of each. Why are these all possibilities?

In case i or ii the reasoning is almost immediately complete.

b. Give the reasoning for these two cases.

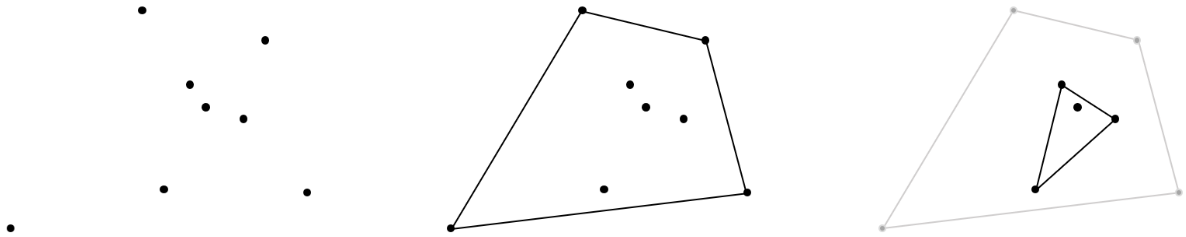
In case iii, the remaining two points lie in the triangle (why?). Call them A and B. Consider line AB.



c. Now complete the reasoning.

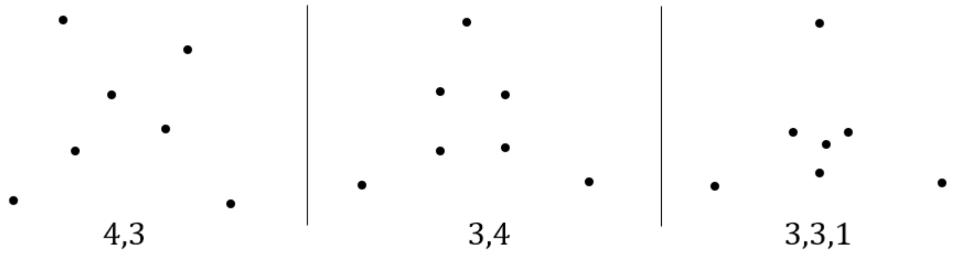
Peeling a point cloud like an onion

You can take the convex hull of a good point cloud. If you then ignore the vertices of the convex hull, the remaining point cloud also has a convex hull. If you then ignore those points again, you can take a convex hull of the remainder, and so on, until there are only 0, 1 or 2 points left. The numbers of vertices of each convex hull then give a sequence of numbers. Below is an example where that sequence is 4, 3, 1.



We call such a sequence the *peeling sequence* of a point cloud.

Below you see examples of each possible peeling sequence for a scatter plot with seven points where the numbers are at most 4.

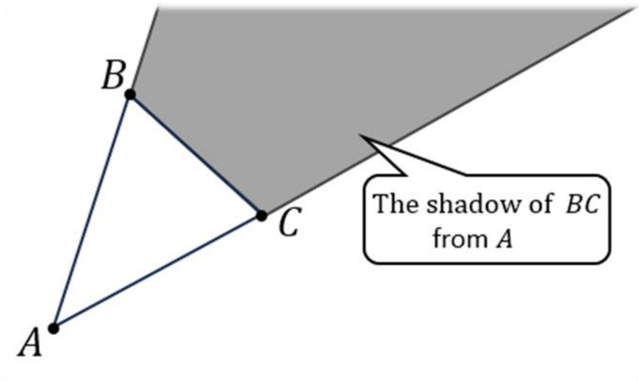


Exploration 3

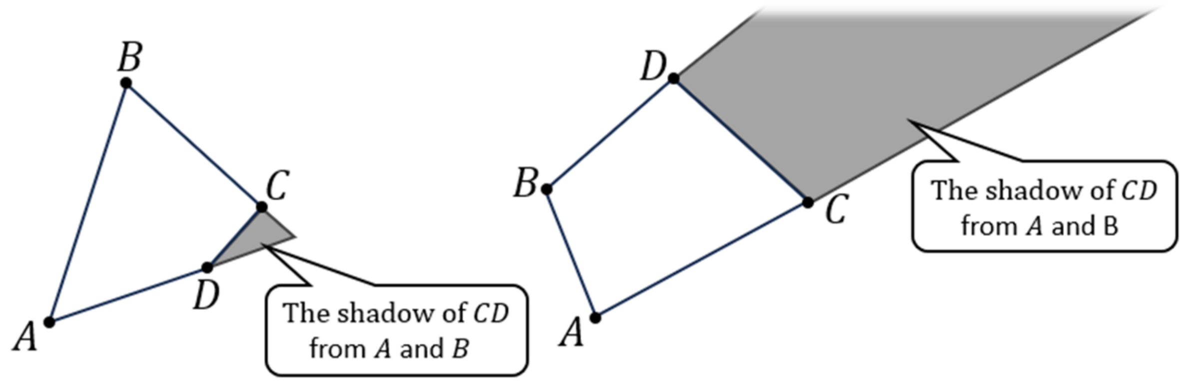
Draw some scatter plots and indicate their peeling sequence.

Shadow

For the sequel it is useful to introduce the notion of shadow. When you have a point $A$and a line segment BC then the shadow of BC from A the area enclosed by line segment BC and lines AB and AC, on the side of BC where A is not located.



You can also look at the shadow from two points: that is the intersection of the shadow from the separate points. In other words, a point is in the shadow of line segment CD from point A and B if it is in the shadow from A and from B.



This area is called the shadow of CD from A and B. Sometimes the shadow is a finite area and sometimes it is infinitely large.

Exploration 4

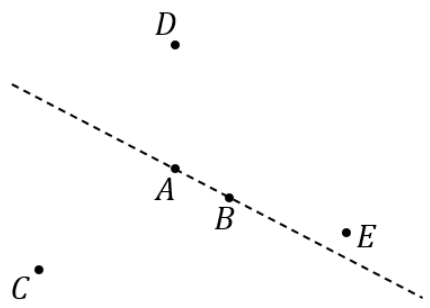
Draw shadows on some triangles and quadrilaterals. Do you see the connection with Exploration 2?

Problem 4

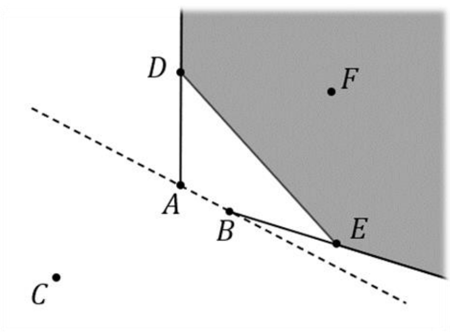
In Problem 3 part c you gave an argument why in case iii (a convex triangle with two points in it) there was always a convex quadrilateral. Now argue that again using shadows. Start with a triangle with one point in it and argue that the second point cannot be placed anywhere without creating a convex quadrilateral.

Problem 5 (Reasoning with shadows and peeling sequences)

We are now going to help you reason that a cloud of points with peeling sequence 3,3,2 always has a convex pentagon. We denote the inner points with A and B. Just as before, lies on one side of AB one point, say C, and on the other side two, say D and E[2]. The inner triangle is therefore CDE. We call the points of the outer triangle F, G and H. We are going to reason about where those points can be located.



If one of the points of the outer triangle lies in the shadow of $DE$from $A$and B, then we have found the convex pentagon.



Now suppose that there is no point there. Then either two points of the outer triangle are in the shadow of CD from A, or two points are in the shadow of CE from B.

a. Explain why (here you can experiment with the app: https://www.fisme.science.uu.nl/toepassing/29255/vrije_verkenning.html).
b. From there, further elaborate the reasoning why a point cloud with peeling sequence 3,3,2 always has a convex pentagon.

[2]We do this so that $ABED$ is a convex quadrilateral (see Problem 3).

In the same way, you can reason that a scatter plot with peeling sequence 4,3,1 always has a convex pentagon. Denote the interior point by A and the vertices of the triangle by B, C and D. Then look at how the four points of the quadrilateral (E, F, G and H) are distributed over the shadows of BC, CD and DB from A.

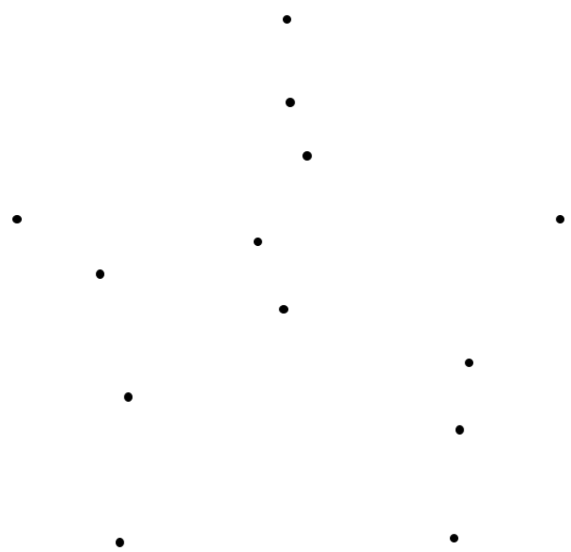
c. Make this idea into a coherent argument.

Hexagons, heptagons and so on...

The game we introduced at the beginning can of course also be played in a variant for convex hexagons, heptagons, and so on. For hexagons, the corresponding question is how long the game can last at most, or:

From what number of points is every good scatterplot with that number of points guaranteed to contain a convex hexagon?

For quadrilaterals, the answer is "five." And for pentagons, the answer to this question is "nine." The image below (and on the front page) shows a good scatter plot of 13 points without a convex hexagon (you don't need to argue for this now).



Exploration 5

Can you make a scatterplot with more than 13 points without a convex hexagon? Experiment in the app:
https://www.fisme.science.uu.nl/toepassing/29255/vrije_verkenning.html (For "N-hoek" top right, select 6).

In general, if $n \geq 3$ is an integer, then you can try to answer the following question:

> *From what number of points is every good point cloud with that number of points guaranteed to contain a convex $n$corner?*

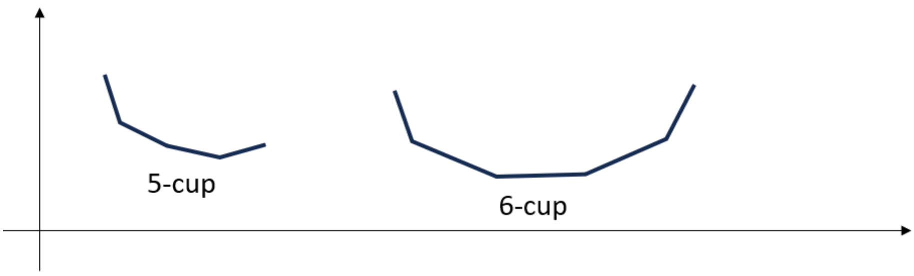
We denote this desired number (provided it is finite) by $C(n)$. We already know that $C(3) = 3$, $C(4) = 5$ and we claimed above that $C(5) = 9$, but the reasoning is not yet complete—that is Choice Research 1. From the figure above you can conclude that $C(6) \geq 14$, the smallest number of points for which a set of six always forms a convex hexagon is not smaller than 14. In other words, the number fourteen is a *lower bound* for $C(6)$. If you have a cloud of points of 14 points without convex hexagons, then you have improved that lower bound.

Would $C(n)$ always be a finite number? For example, it is conceivable that you can add infinitely many points without ever getting a convex hexadecagon. In fact, one can prove that $C(n)$is a finite number for every n, but that goes a bit too far for today. If you have anything sensible to say about it, please feel free to add it to your report.

One way to systematically improve the lower limit is with so-called cups and caps.

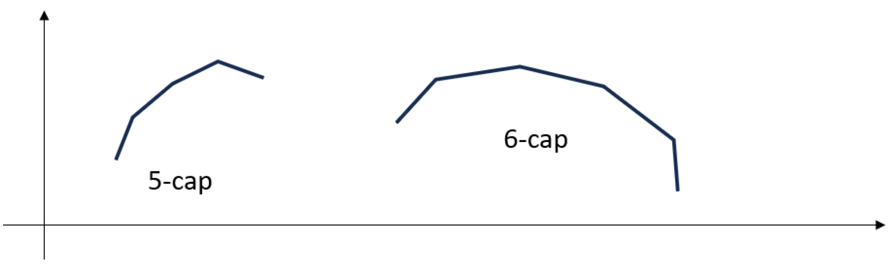
CUPS AND CAPS

A *chain* is a finite series of line segments, each segment beginning at the vertex where the previous one ends. A chain *in a coordinate system* is a *cup* if every vertex is to the right of the previous one and each successive segment has a greater slope (gradient) than the previous one.



We write n-cup, where n stands for the number of vertices.

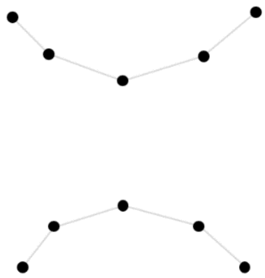
Conversely, a chain *in a coordinate system* is a *cap* if each point is to the right of the previous one and each subsequent line segment has a smaller slope (gradient) than the previous one.



Cups and caps can help create point clouds without convex polygons.

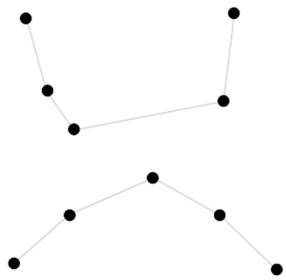
PROBLEM 6

Place dots in a 5-cap horizontally opposite dots in a 5-cup as below.



a. Reason why the above scatter plot of 10 points does not contain a convex hexagon.

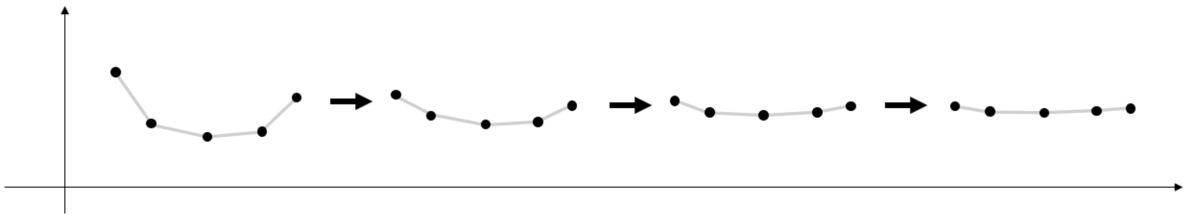
A 5-cap horizontally opposite a 5-cup sometimes contains a convex hexagon.



b. Explain under what conditions in general a 5-cap horizontally opposite points in a 5-cup does not contain a convex hexagon.

c. Generalize your statement to an n-cap opposite an n-cup for a $n = 2, 3, 4, \ldots$What is your conclusion for lower limits for $C(n)$?

Of course, there are still some points that fit in here, between the cap and the cup. To keep track of all those points, it is useful to almost flatten the caps and cups. In doing so, you only reduce them in the vertical direction by a factor.



d. Explain that in this way, cups remain cups and caps remain caps, but the slopes can be made as close to zero as you like.

e. Try to place flattened caps and cups cleverly on the middle dotted lines below to create a larger point cloud without a convex hexagon. Test your design in the app if necessary.

Own research

We invite you to choose one (or more) of the topics below for your own research.

As a reminder, the research paper consists of an introduction to point clouds, convex polygons, and the central problem, based on your findings on the basic problems. Then you will describe your approach and results in at least one of the investigations that you choose from the options below.

Investigation 1 (Reasoning for convex pentagons)

Using peeling sequences and by further developing what you did in problem 5, you can reason that a good scatter plot of 9 points is guaranteed to contain a convex pentagon. Give such a argument. Also, give an example of a good scatter plot of eight points without a convex pentagon.

Tip: For a scatterplot of nine points, describe all possible peeling sequences where the numbers are at most 4. Then give a argument for each peeling sequence, if you have not already done so in Problem 5.

Research 2 (Making lower limits with headings and caps)

The next challenge is to create the largest possible point cloud (i.e., with as many points as possible) without a convex heptagon.

For this, it helps to make, in addition to regular heads and caps, scatter plots that contain at most a 3-cap and a 3-cup, but no 4-cap or 4-cup. We call such a scatter plot 3,3-cop.

a 3,3-cop

Not a 3,3-cop

The right one is not a 3,3-cop, because a 4-cup can be made. However, a 3,3-cop can be made with more than 4 points.

a. What is the 3,3-cop with the most points that you can make?

In general, a k,l-cop is a point cloud with at most a k-cap and a l-cup, but no $(k+1)$-cap or $(l+1)$-cup.

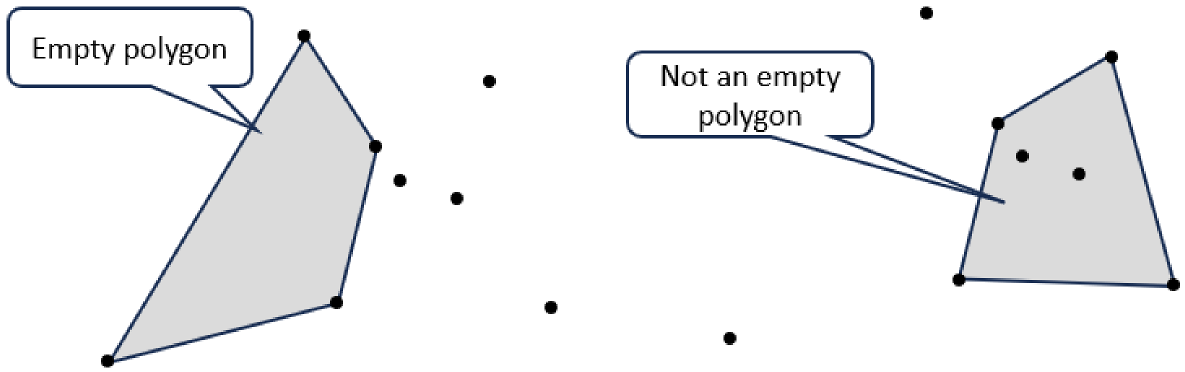
b. Investigate using methods from Problem 6 what is the largest point cloud you can make without a convex heptagon. Hint: it is definitely possible with more than 25 points. You can also use the app here.

Reminder: $C(n)$stands for the smallest number of points in a good point cloud where there are always a n-number of points forming a convex n-angle

c. Can you find a relationship between $n$and $C(n)$? This is not about your hypothesis, but about how you arrived at it.

INVESTIGATION 3 (EMPTY CONVEX POLYGONS)

In a good point cloud, a polygon is *empty* if there are no points within the polygon.



We can slightly modify the central question we asked earlier:

> *From what number of points is every good point cloud with that number of points guaranteed to contain an empty convex n-corner?*

We denote that smallest number with $LC(n)$.

a. Modify the argument from Problem 3 to reason that $LC(4) = 5$

b. Research: What is $LC(5)$? In other words: What is the smallest number of points in a good scatterplot where there are always five points that form an empty convex pentagon? Support your answer with arguments. Hint: $LC(5) \neq 9$.

INVESTIGATION 4 (CONVEX HEXAGONS)

How many points in a good scatter plot guarantee a convex hexagon? In Problem 6, part e, you found a lower bound of $C(6)$. Can you also reason that this is an upper bound, or that a convex hexagon is guaranteed to be formed with more points? How far do you get if you reason in the same way as in Problem 5 with convex pentagons?