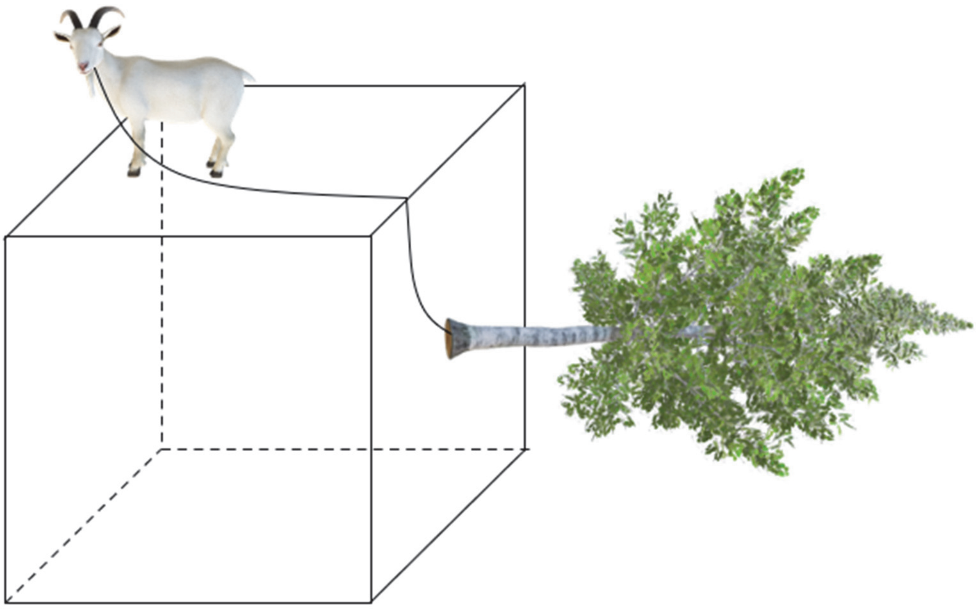
Geometry on the surface



Mathematics B-day 2021

Universiteit Utrecht

Wiskunde voor teams

Freudenthal Institute

Introduction

About the assignment

For most people, geometry begins in a two-dimensional flat plane. There, lines intersect or run parallel. The set of points equidistant from another point is a circle. The angle sum of a triangle is 180 degrees. If all is well, then those are known facts for you. Mathematicians have been studying these properties on non-flat (curved) surfaces since ancient times. For example, you may have heard of spherical geometry – geometry on a spherical surface, such as the surface of the Earth. This is a type of 'sport' that is relevant for satellite orbits and for how earthquake vibrations spread. Today you are going on a similar adventure. You are going to do geometry on the surface of, for example, a cube, a beam, or another polyhedron. That mathematic is important for understanding vibrations across a building or calculating distances across its surface. You can still draw a circle on a sphere, but what about polyhedra? Things turn out to be surprisingly different!

Structure of the day

This Mathematics B-day assignment consists of introductory tasks and additional, in-depth problems. Unlike in normal math lessons, you certainly don't have to solve all the problems in the math B-day. If you get stuck with a problem or don't have enough time, you can let it rest or even skip it completely. To help you on your way, there are suggestions for every problem. Problems range from easy to hard, so it's normal if you don't get everything done; but **at least show in the report what you have tried and how far you have come with problems 1 to 5 – for example by using the suggestions**. Problems 6, 7 and 8 are additional. After you have spent enough time on problems 1 through 5, choose one or more of the additional problems to delve deeper into a topic. With success on these last problems, your team can distinguish itself even more!

Working in teams

The special thing about the mathematics B-day is that you do mathematics as a team. It may be a good idea to make a schedule and a division of tasks. Let everyone do what they are good at. Give everyone space to contribute with ideas and elaborations. You can all work on different tasks at the same time, or work together on a problem, and then come together again to discuss and evaluate. For some problems it is useful to study some different examples. That is something that can easily be divided with your team.

Supplies

Today, you will need: a pen, enough (scrap) paper, scissors, tape and/or glue, this assignment, and a computer or laptop to prepare your report. We want to discourage the use of the internet; if you do use online sources, include a citation in your report (url).

What to hand in?

You will work on a digital report during the day. Don't start too late with that. You must hand it in at 16:00. In it, you describe your results and reasoning. Tell your own, clear and convincing story. We appreciate well-written, clear, precise, complete, carefully formulated, and certainly original, creative and lyrical reports.

Tips:

- It can be useful to start writing out your results in the morning already.

- *Be understandable*: make sure that the text is legible for someone who did not take part in the Mathematics B-day (but who does have sufficient understanding of mathematics), *without having read the assignment*. You do not have to literally copy the problems from the assignment in the report. Instead, make it a running story.
- If you provide substantiations, explanations or clarifications, try to do so *with mathematical arguments* as much as possible.
- Use *figures* to illustrate your ideas. For example, use copies of pictures you have made (screen captures or photos of figures on paper).
- Make a *schedule and divide the tasks* among the team.

Both the mathematical content of the report and the way it is written will count in the assessment!

Introductory problems

Problem 1 (Goat tracks on a cuboid)

On a (rectangular) cuboid with dimensions $4 \times 4 \times 10$ points P and Q lie on the left and right 4×4 face, respectively. Point *P* is 3 below the center of an edge in the top face and point *Q* is 1 below the center of a edge in the top face (see Figure 1). A goat wants to walk across the surface of this cuboid from point *P* to point *Q*. The shortest path is not the route below of length 14. Investigate: can you find a shorter path? How long is the shortest path you can find? Why do you think it can not be shorter?

Suggestions

- Make different nets of the cuboid and fold it in and out. Put the different results in the report.
- Look for the shortest paths on the nets.
- You can read ahead to the text under this problem if you like.
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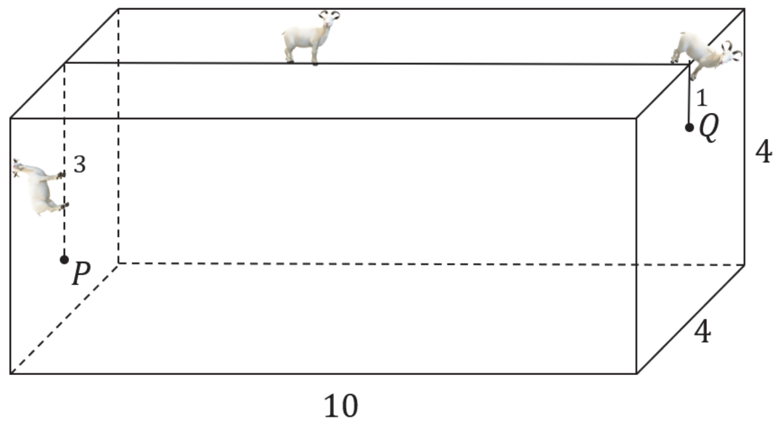


Figure 1. A goat walks across a beam

Aside: Take a strip of paper and draw two points *P* and *Q,* and the line segment *PQ* on it. Fold the paper along a line that intersects segment *PQ*, as in Figure 2. The intersection of segment *PQ* with the fold we call *S*. Draw another point *T* on the fold as well. The line segments *PS* and *QS* together seem shorter than *PT* and *QT* together, right? Of course, this is the case, because *PS* and *QS* lie together on segment *PQ*, which is the shortest path between *P* and *Q* in the plane.

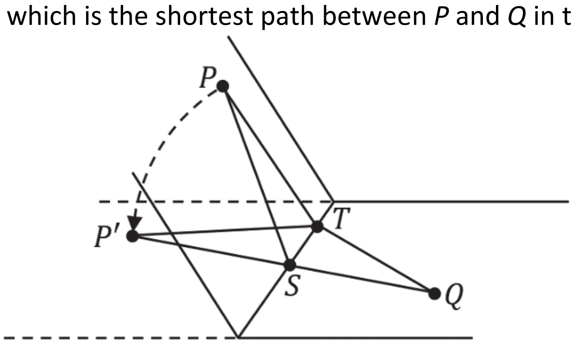
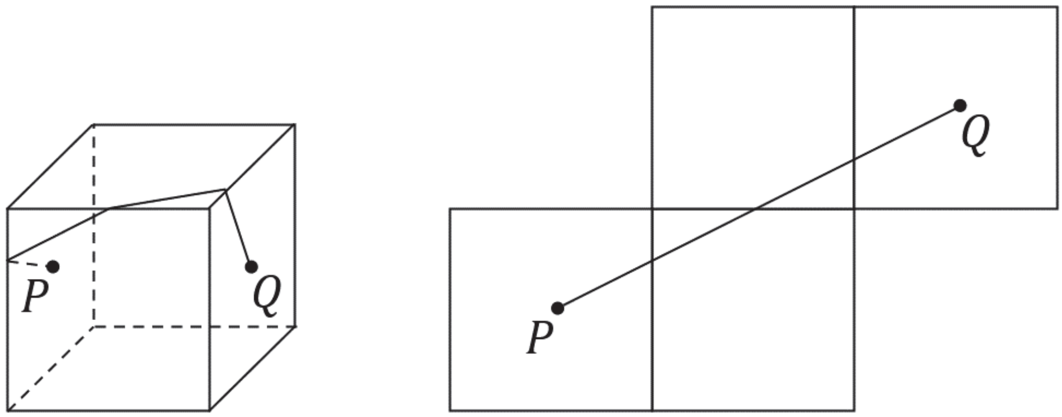


Figure 2. A line segment PQ intersecting an edge is straight (P'Q) when folded out flat

Even if a path between two points is straight at each edge, if you unfold the polyhedron there, it still does not guarantee that the path is also a **shortest** path between those points. That was the lesson of problem 1: there can be many paths between two points that are straight when folded flat at every edge (and are straight at every face), but still not be the shortest path.

A **line (segment) on a polyhedron** is a path that is straight at any face, and if the path intersects an edge, then the segments on the faces when folded flat along that edge must be in line. A line (segment) on a polyhedron cannot pass over the vertices of the polyhedron. The shortest path between two points on a polyhedron is a line segment, but vice versa, not every line segment is a shortest path (and that is very different from in the flat plane!)

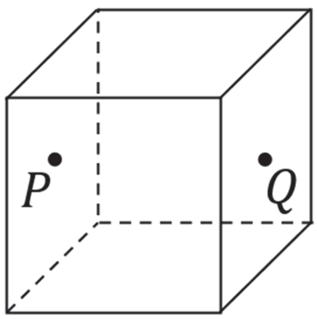


Figuur 3. Left: A line segment on the cube - but not the shortest path. Right: the same line segment in a part of the net

Problem 2 (Investigation of line segments on a cube)

a. Investigate: Find a line segment on the cube that bisects itself (suggestions on the next page).

b. Point *P* lies in the center of the left-hand face of a cube and point *Q* directly opposite it at the center of the right-hand face. Investigate how many line segments run from *P* to *Q*. Note: a line segment does not have to be a shortest path (see above).



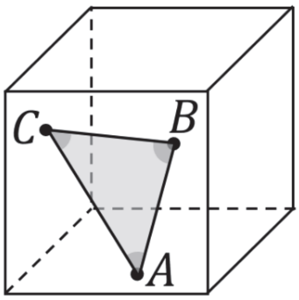
c. Why should we agree that a line segment cannot pass through a vertex?

Suggestions:

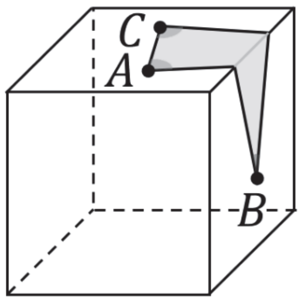
- Make different nets of the cube, place them against each other if necessary, fold them open and close again, and look for different possibilities.
- Important question: can a line segment cross the same face more than once?
- If you have an idea, also try to give reasons why it is true; why can't there be more?

A triangle on a cube, like in the plane, is an area enclosed by three points, connected by three segments on the cube (which do not intersect themselves and each other).

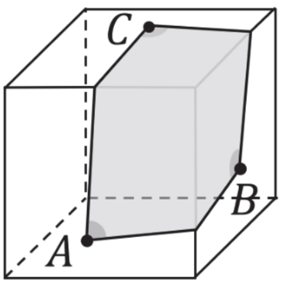
Below, you see four examples of triangles *ABC* (in Figure 4), i.e., the gray area enclosed by line segments on the cube *AC*, *AB* and *BC*. It's like pulling a rubber sheet tight over the cube and securing it at points *A*, *B*, and *C*.



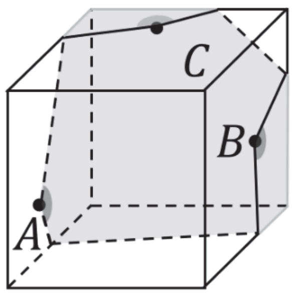
Triangle on the front face



Triangle “folded around an edge”



Triangle “stretched around the top right front corner”



Triangle “stretched around **three** vertices at the back”

Figure 4: Four examples of a triangle on a cube

You see three angles at points *A*, *B* and *C* (we ignore the angles at the edges). Normally, the sum of the angles of a triangle is 180 degrees. This is clearly not the case with the bottom two triangles, so there is something going on here.

Problem 3 (The angle sum of a triangle on a cube)

Investigate: what is the sum of the angles of a triangle on the cube? Explain! (Suggestions on next page)

Suggestions:

- The angle sum is not always the same. But there is a nice relation you could formulate.
- Examine at least six examples of triangles on the cube that are as different as possible using different nets.
- Formulate hypotheses about the angle sum and perhaps test them in new examples.
- If you have found a relation, give reasons why it is so. Examples and figures can help explain, but your arguments should be more generally valid.

Problem 4 (Cubic Goat Pasture)

Delicious grass grows on all faces of a cube with edge 2. In the middle of an edge, there is a tree. The farmer has tied the goat to the tree with a rope. The shortest rope with which the goat can reach all the delicious grass is longer than $\sqrt{13}$ and shorter than $\sqrt{17}$. What is the shortest length for the rope that you can find? Why do you think it can not be shorter?

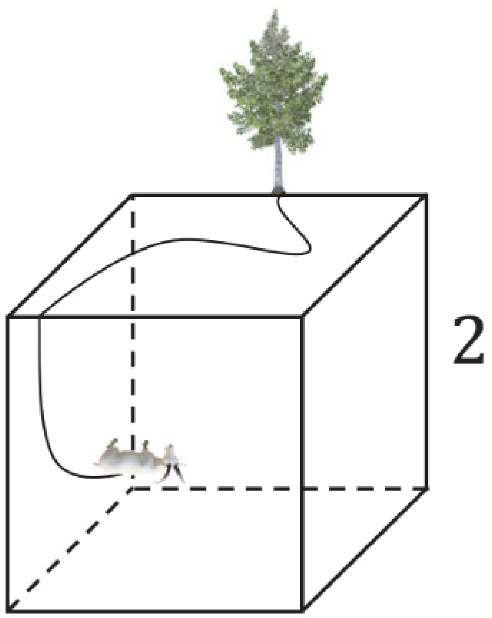


Figure 5. A goat on a rope to a tree on a cube

Suggestions:

- First, explain why the most distant points are **not** the vertices of the cube, by finding at least one point that is further away. The following two suggestions can help with this.
- Draw the situation from a different direction.
- An exploratory study can consist of making the rope longer and longer and seeing which points you can reach with a tense rope. With length 1 you just reach the two nearest vertices and your range consists of two semicircles. Then it gets more complicated…
- A most distant point can be reached from the tree along two (equally long) shortest paths. Why?
- If you have an idea, name a cleverly chosen unknown distance *x* and continue with algebra.
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If the rope remains tense, the goat walks in a **circle on the polyhedron**, in the sense of “points at a fixed distance from the center”. Circles on a polyhedron appear to consist of several arcs of circles glued together.

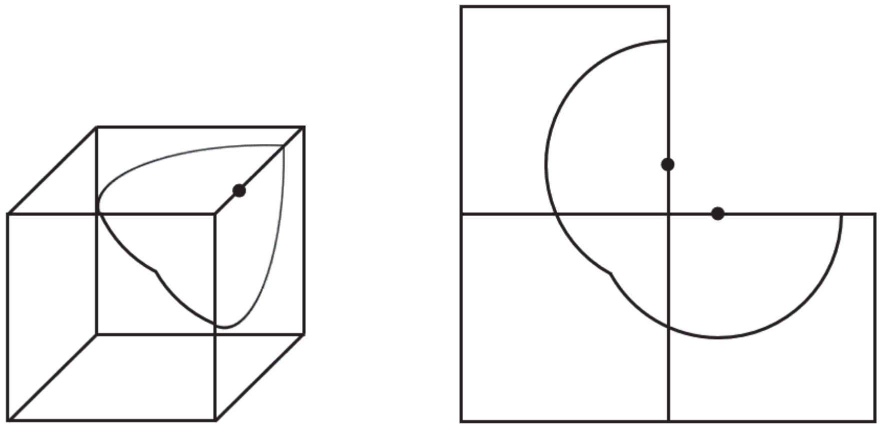


Figure 6. A circle on a cube. See the arcs on the interfaces (left) and in (part of) the net (right)

Problem 5 (Three grazing goats)

A farmer has three goats, which are allowed to graze on the juicy grass on the boundaries of the cube. The goats are each tied to a post with a rope. Each rope is the same length. Investigate: where should the farmer position the poles and how long should the ropes be, so that as much grass as possible can be reached, but the goats can't touch each other? Present the best solution you can find in the report; explain how you found it; and possibly explain why you think it couldn't be better (suggestions on the next page).

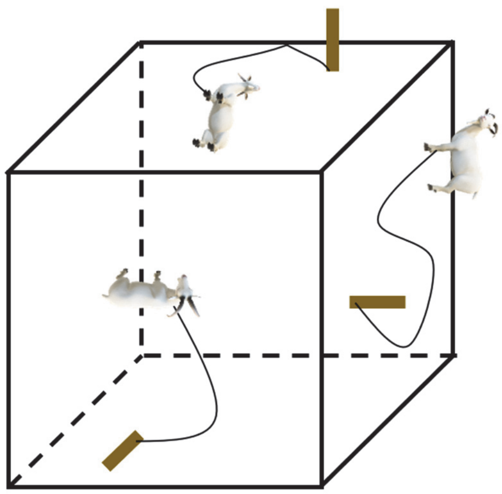
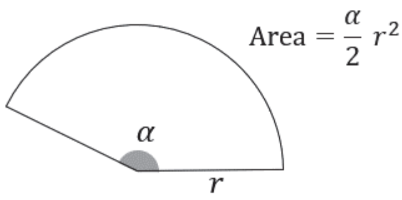


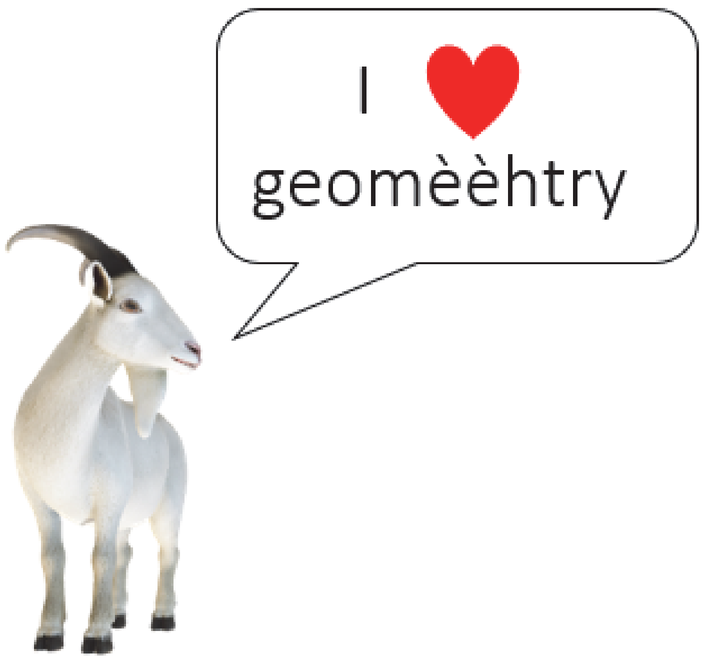
Figure 7: three goats grazing on a cube

Suggestions

- In any case, try to explore a number of different options
- Can you think of arguments why a solution must have certain properties?
- Calculate, of course, the length of the rope and the surface of the area. The area of a circle sector with angle α (in radians) and radius r is $\frac{\alpha}{2\pi} \cdot \pi r^2$ (why?), which can be simplified to $\frac{\alpha}{2} r^2$.



To calculate the area of a circle on the cube you often have to divide the figure into circle sectors and triangles (Area $= \frac{1}{2}$ base $\cdot$ height).



Additional problems

We invite you to pick one (or more) of the problems below, which are a further exploration of the introductory problems.

In Problem 1 you struggled with finding a shortest path between two points on a polyhedron. Perhaps the search was somewhat inefficient and chaotic then. Now, let's look at the same problem on a regular tetrahedron (polyhedron with four equilateral triangles as interfaces, a so-called tetrahedron).

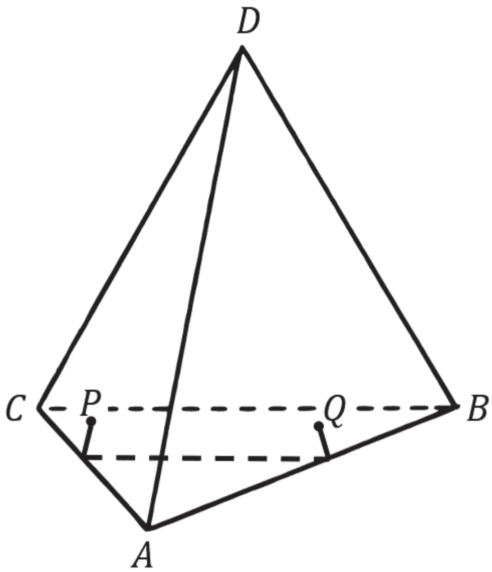


Figure 8: A regular tetrahedron. Is the dashed path the shortest connection between P and Q?

The shortest connection can be found efficiently using a triangular grid which can be seen as a field of foldouts that have been glued together.

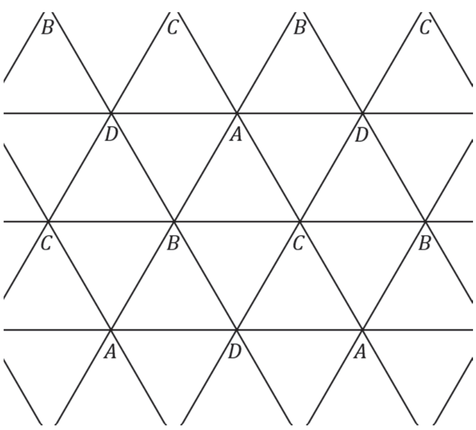


Figure 9: a field of glued together foldouts of a regular tetrahedron

Problem 6

a. Explain how to use the grid in Figure 9 to efficiently and effectively find the shortest connection between *P* and *Q*.

For other polyhedrons, it is not so easy to stick the nets together, unfortunately. The challenge now is to find other efficient and structured ways to find and describe line segments and the shortest connection between two given points.

b. Try this for the cube first. Then extend your method to other polyhedrons if you want to.

Problem 7

In Problem 5 you tried to place three goats on a cube. Can you solve this same problem on a different polyhedron and/or with a different number of goats? Find yourself a combination of polyhedron and number of goats where the problem is interesting and challenging.

Problem 8

In problem 3 you investigated the angle sum of a triangle on a cube. What about the angle sum of other polygons? There are even so-called digons on a cube (with two vertices and two sides) and monogons (with one vertex and one side). Try to extend your insights to all the polygons on the cube. In addition, you can try studying angle sums of polygons on other polyhedrons (of your choice, for example the regular tetrahedron from Problem 6).

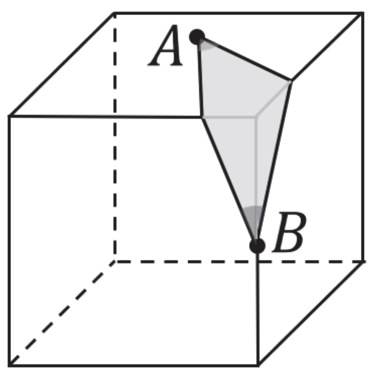


Figure 10: A digon on a cube