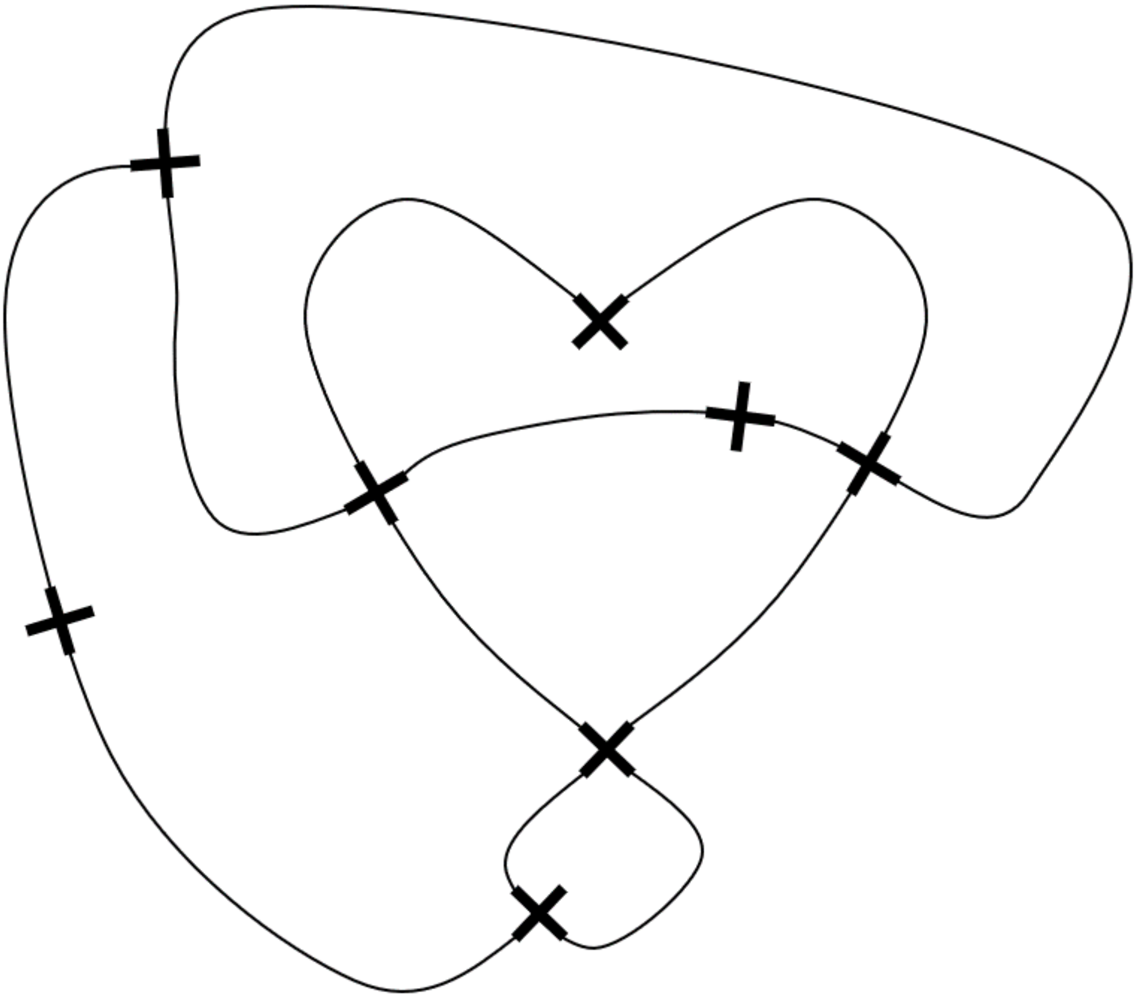
Connect and conquer



Mathematics B-day 2019

Universiteit Utrecht

Wiskunde voor teams

Freudenthal Institute

Introduction

About the assignment

This Mathematics B-assignment is about something we all love to do: winning games. Sometimes that is down to luck, but not with the game you'll be playing today…

Working in teams

The special thing about the Mathematics B-day is that you are doing mathematics in a team, as you would do with for example football. It may be a good idea to make a planning and divide tasks. Let everybody do what they're good at, and give everyone room to contribute with ideas and elaborations.

Structure of the day

This Mathematics B-day assignment consists of basic assignments, and the final assignments. Try to spend approximately half the day on the final assignments.

Necessities

Today you will need: a pen; enough (scratch) paper; scissors, glue, sticky tape, a stapler or paperclips to attach sheets of paper to each other, and a computer or laptop to make your report on. Using the internet is allowed (clearly mention your source-url in your report), but not encouraged, since we are primarily interested in your own work.

What to hand in?

You will work on a digital report throughout the day. Do not start too late, because you have to hand it in by 16:00.

In the report you describe your results and reasonings – even if they are not wholly complete. The focus is on the research in the final assignment.

Tell your own, clear and convincing story. We appreciate well-written, clear, precise, complete, carefully formulated and certainly also original, creative and lyrical reports. Both the mathematical content of the report and the way it has been written down will play a part in the assessment!

Tips for writing and lay-outing the report:

- It may be useful to start writing down the elaborations of the basic assignments in the morning already.
- *Be clear*: someone who did not take part in the Mathematics B-day (but who does have a sufficient understanding of mathematics) should be able to understand it.
- When you are providing underpinnings, explanations or clarification (and this is important), make as much use as possible of *mathematical arguments*.
- Use *your own figures* to illustrate ideas. Use, for instance, copies of images made by you (screen-captures or photos of figures on paper).

Basic assignments

Today you will work with the game *Connect-and-conquer*. Different variants of the game will be used, so pay attention to which variant the assignment is about. The first version of the game starts with two +-signs (see Figure 1).

Figure 1. A starting figure for the game.

You can look at every +-sign as a dot with four tails. We call that a 4-node. So in total there are eight tails in the starting situation in Figure 1.

Rule

Two players make a move in turn. A *move* consists of connecting two different tails (from the same node or from two different nodes) in which you place a new +-sign on the connecting line (which you draw without lifting your pen from the paper), that is to say a new tail on each side and the other two tails exactly on the connecting line.

You end up with for example:

Figure 2. An exemplary first move in *Connect-and-conquer*

In the next move, again two unused tails are connected and a +-sign is placed on the connecting line; we get for example

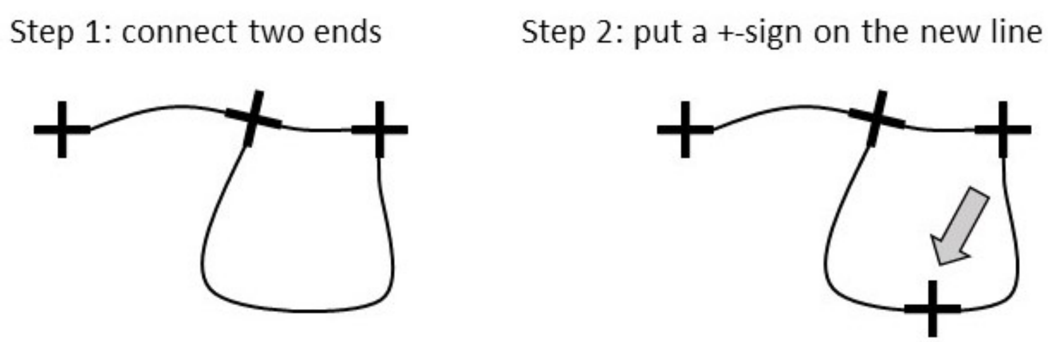


Figure 3. An exemplary second move in *Connect-and- conquer*

Like this, two unused tails may be connected each time through a connecting line – keeping in mind that **the connecting line may not cross or cut across an existing line, node or tail**. So the tail indicated with  in Figure 3 will not ever be connected to another tail. Check this!

The first player who cannot make a move because it is no longer possible to connect two tails without crossing a previously drawn line, has lost.

The starting nodes can vary in the number of tails, see below. You still always put a single perpendicular line on new connecting lines, so that you get a +-sign.

Exploration. Play the game a few times with the different initial figures below. You do not have to put the results of these (and later) explorations in the report.

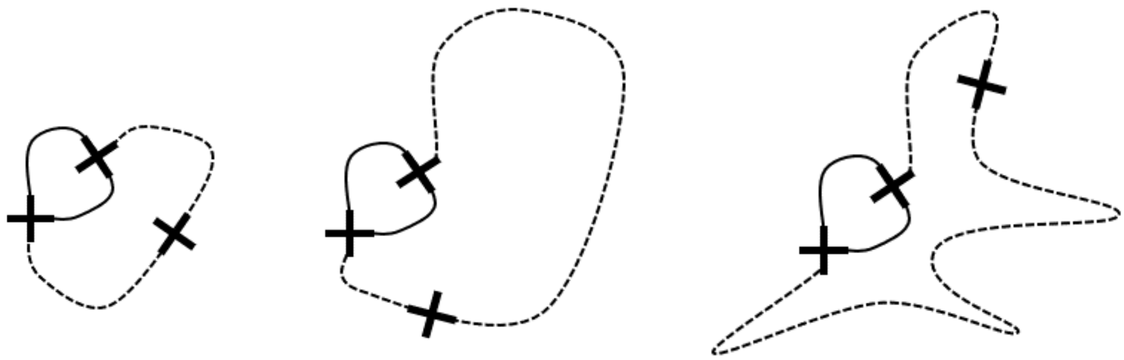
Two 4-nodes One 8-node A 3-node and a 4-node

It looks as if the game always stops of its own accord, and that, no matter how you play, you have to play the same length of time for each starting figure. The challenge for you is to find out a lot about this game by experimenting and reasoning.

Let's begin with researching a simple situation: starting an initial situation with one 4-node. There are not many ways for the game to develop.

Assignment 1 (overview). Give a well-considered overview of all the ways that the game can run with one 4-node in the initial situation.

The overview you make in assignment 1 will not be infinitely large because you do not distinguish between for example the moves indicated below with dotted lines.



It seems to be irrelevant for how the game plays how *exactly* the lines run and where the new node and the tails end up *exactly*.

Assignment 2 (moving around lines and nodes). Research whether it matters for the gameplay if you move around a line or a node (that was drawn earlier in the game) without intersecting other lines and nodes, and provide arguments for your findings.

We now consider initial situations with just one node. Instead of one 4-node or 8-node, we can also consider in general a game that starts with a d-node: a node with d tails. Can you determine in advance for each value of d who will win? The challenge for now is to answer this question and of course understand the answer.

For this it turns out to be handy to look at *areas* in the game. At the start of the game your image exists of one *area*: the whole plane on which you play the game. In Figure 4 you can see a stage of the game with four areas.

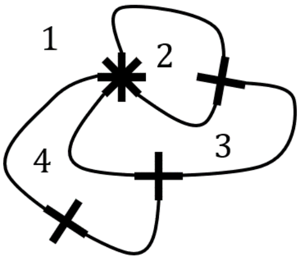
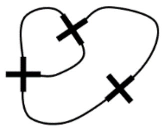


Figure 4. A game stage with four areas and eight unused tails.

During each stage of the game there are *used* and *unused* tails. The unused tails are the ones without a line departing from them.

Assignment 3 (research with chart). Use a chart to research what happens to the number of unused tails and the number of areas during games that start with a single d-node and explain what you see.

Game situation	Move number z	Number of unused tails d	Number of areas g
	0	4	1
	1	…	2
	2	…	…
…	…	…	…

The number of unused tails d is a so-called *invariant*: this number doesn't change. The move number z minus the number of areas g, i.e. $z - g$ is also an invariant.

If you consider the move z, the number of areas g and the number of unused tails d at the end of the game, you will see that the situations are actually compatible.

Assignment 4 (a formula for the number of moves).

- **a.** Give a formula for the number of moves for a game that starts with one node expressed in d and use it to explain who wins, the first player or the second player.
- **b.** Explain why your formula is true. As part of this, explain why the game stops after so many moves, and why there is always at least one possible move up to then.

First mission accomplished! But we are not yet done. This was only for games that start with just one node. What about games that start with more nodes?

From now on, we will look for a while at games that start with just 4-nodes. The number of 4-nodes in the initial situation is referred to with n.

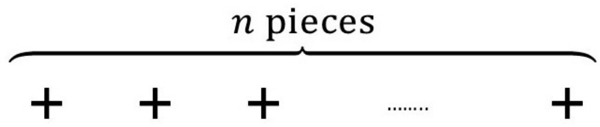


Figure 5. The new initial situation with n 4-nodes

Exploration. Play the game for $n = 3$, that is with three 4-nodes at the start. Is your equation from Assignment 4a still true? Can you predict who will win for this initial situation?

If you start with more nodes, you will see while playing that nodes are connected in different groups. We call those *components* (see Figure 6) and denote the number of components with c.

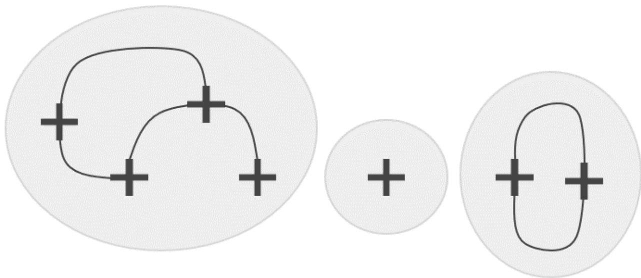


Figure 6. Example: three components (during a game that started with four 4-nodes)

During the game we connect two unused tails in each move. If those two tails belong to two different components, the number of components is reduced by one. We call this a *connecting move*. If those two spines belong to the same component, an area will be split in two and we call this a *splitting move*.

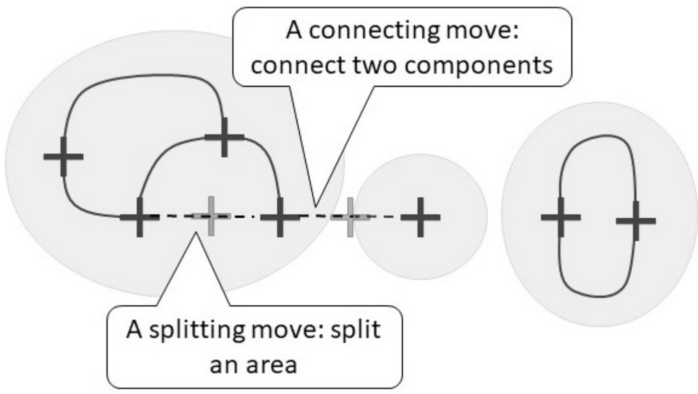
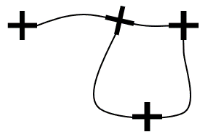


Figure 7. An example of a connecting and a splitting move

Assignment 5 (research with chart II). Fill in the chart below up to and including the last move for a game with two 4-nodes and for a game with three 4-nodes. Feel free to use your own game instead of the one shown.

Game situation	Move number z	Splitting/ Connecting move	Number of unused tails d	Number of components c	Number of areas g
	0		8	2	1
	1	V	8	1	1
	2	S		…	…
…	…	…		…	…

It looks as if the number of moves is also fixed for an initial situation with more than four 4-nodes.

Assignment 6 (a formula for the number of moves II).

- **a.** Give a formula for the number of moves in a game with n 4-nodes and use it to explain who wins.
- **b.** Explain how you get your answer for **a**. You can use for example your observations about
  - The number of connecting moves and splitting moves
  - When the game continues and when it stops (pay attention to the number of tails and the number of areas)

Assignment 7 (reasoning with an invariant)

Explain why $z + c - g$ is an invariant (that is to say: the formula has the same outcome for every move, doesn't vary). To do so, look separately at what happens in a splitting move and a connecting move. Explain the formula from Assignment 6 part (a) with the help of this invariant.

Exploration. Play/Research some games now that have an initial situation with different types of knots together, as in Figure 8 for example. Can you predict the number of moves based on just the total number of tails d and the number of nodes n?

Assignment 8 (n nodes). Research who wins the game if you start with n nodes that may have different numbers of tails, say $d_1, d_2, d_3, \dots, d_n$. Look at an example in Figure 8. Explain clearly how you found your results.

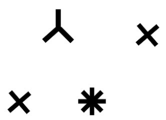


Figure 8. An initial situation with $d_1 = 4$, $d_2 = 3$, $d_3 = 8$ en $d_4 = 4$. Who wins?

Final assignments

After the previous basic assignments, here now follow three possible final assignments. Look all three through globally first, and then choose at least one of the three with your group. But you can choose to do more than one.

Final assignment option 1: Connect-and-conquer with wild card move

This morning you saw that it is certain from the start which player will win in *Connect-and-conquer* – whether or not that player likes it. In this final assignment you will research a variant of *Connect-and-conquer* where the players' moves do matter and it becomes more of a real game.

The rules are the same as in regular *Connect-and-conquer*, except that there is a wild card move in the game: at one point one of the players may make a move in which no +-sign is drawn on the new line. The players may decide themselves if and when they will play this wild card, but once the wild card move has been played, both players can only make regular moves.

Assignment 1 (choices do matter). Show that the choices by the players in this game do in fact influence who wins by

(a) Giving an example of a game play where player 1 wins and
(b) An example of a game play with the same initial situation where player 2 wins.

Assignment 2 (who wins?). Research for as many different initial situations as possible who will win this game if both players play as well as possible. Give your reasonings!

The game gets even more interesting if both players can do one wild card move (or one player two wild card moves, if the other player does not grab his chance at a wild card).

Assignment 3 (two wild card moves). Research for this variant as well for as many different initial situations as possible who will win this game if both players play as well as possible. Give your reasonings!

Final assignment option 2: Surfaces

In this final assignment you will continue a little longer with the game that you played before, but now with a typical mathematical twist.

Roll a sheet of paper into a cylinder and secure it (with glue, sticky tape, staples or paperclips), see Figure 9.

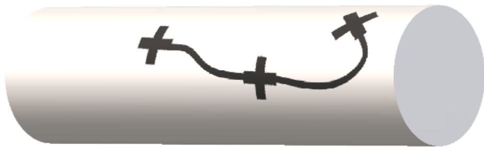


Figure 9. Play the game on a cylinder

Exploration. Play the game on the outside of the cylinder, and freely choose the initial situations. You are *not* allowed to go to the other side of the paper via the edge of the sheet. Research if the game runs differently than on a flat sheet of paper, as you had until now.

Next, make two cylinders that are roughly the same size. Stick them together like in Figure 10.

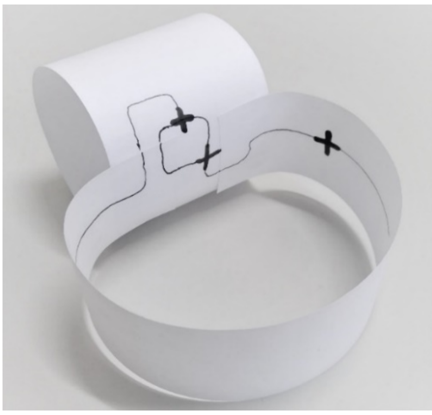


Figure 10. Connect-and-conquer on two connected cylinders.

Exploration: Play the game a few times with one 4-node on the connected cylinders. Is the winner determined in advance?

Sometimes you can make a move on the connected cylinders that is neither a connecting nor a splitting move. We call that type of move a top-move (short for *topological* move). Because of the top-moves, $z + g - c$ is no longer an invariant (as with Connect-and-conquer on a flat surface). It looks as if you have to look more carefully whether there are possible top-moves in an area. Write g_0 for the number of areas where there are no (more) possible top-moves; g_1 for the number of areas where there is one (more) possible top-move; g_2 for the number of areas where there are two (more) possible top-moves, etc.

Assignment 1. (Winning on the double cylinder)

- **a.** Research who wins on the double cylinder if you start with one 4-node.
- **b.** Also research who wins in a game with n nodes with different numbers of tails around it, say $d_1, d_2, d_3, \dots, d_n$, for the largest number of possible combinations.

But now the game is unleashed. How will the game go if the area is extended with one more loop (see Figure 11)?

Figure 11. Connect-and-conquer on three linked looped strips.

Assignment 2 (Tasks with even more twists). Research who wins in a game on a surface with l loops that have been linked like with n 4-nodes, or optionally also with different numbers of tails, say $d_1, d_2, d_3, \ldots, d_n$.

Final assignment option 3: Valley deep, mountain high

In this third possible final assignment, we introduce a new problem which appears unrelated to the Connect-and-conquer game, though you will need a lot of similar reasoning.

The goal of this part is that you discover a mathematical connection between the number of peaks, valleys, saddle points, lakes and islands in a landscape.

Rather than with three-dimensional images of landscapes, it is easier to work with contour images, i.e., height maps.

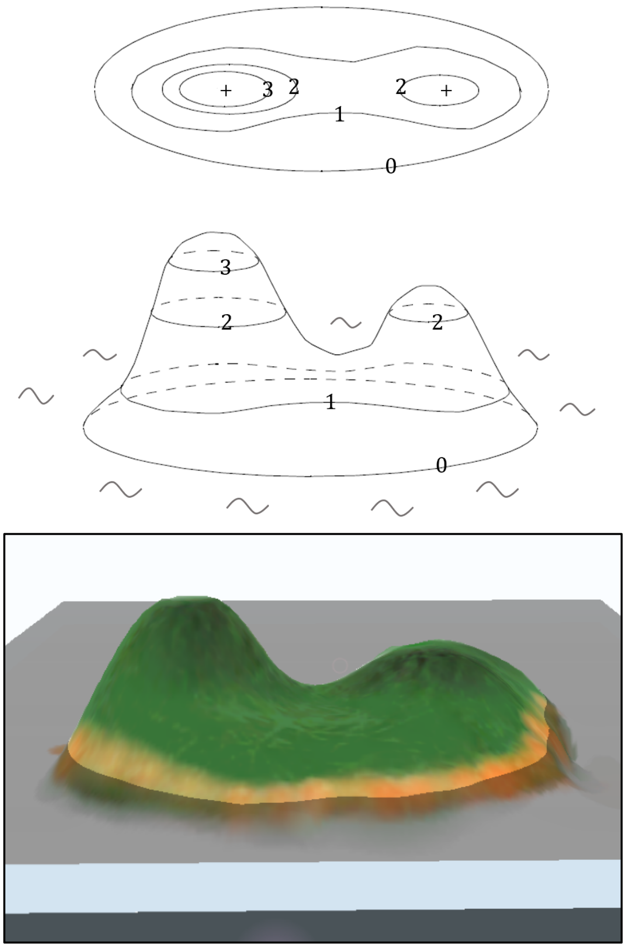


Figure 12. A height map, sketch and image of a landscape.

In the height map there are two points where the surface has been horizontally marked with a plus: the two tops. But there is another point where the surface is horizontal, which can be found more or less in the middle between the two tops. Here four lines (on the (red) figures of eight in Figure 13) at the same height meet, and in between the mountain goes alternately up and down when you move away from the point. This kind of point is called a saddle point because that shape is also used in saddles.

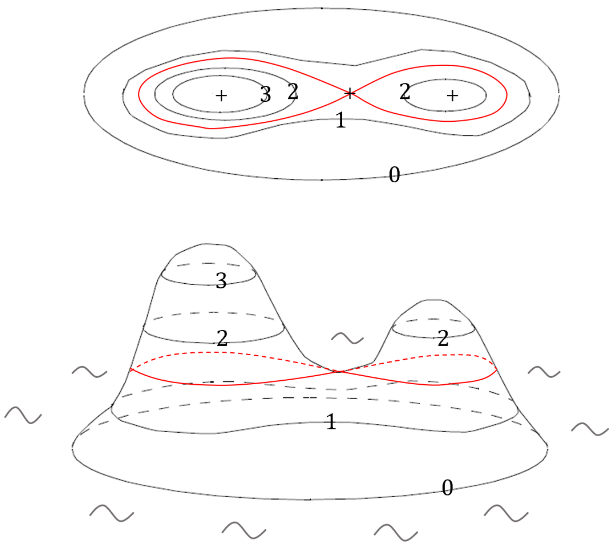


Figure 13. In between the two tops there is a saddle point. In that point, four lines at the same height meet.

Like on earth, the landscapes we are looking at are in the middle of a large ocean. Before we start reasoning with this, first two exercises about landscapes and height maps. The result of these exercises doesn't have to be in the report.

Exercise 1. Sketch the landscape for the height map in Figure 14.

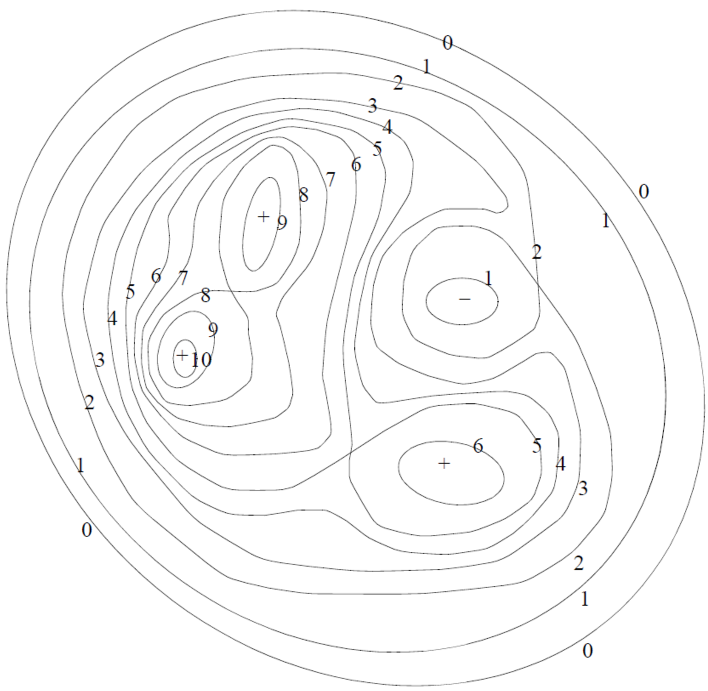


Figure 14. A height map. The minus sign denotes a valley.

Exercise 2. Sketch a height map for the following landscape in Figure 15.

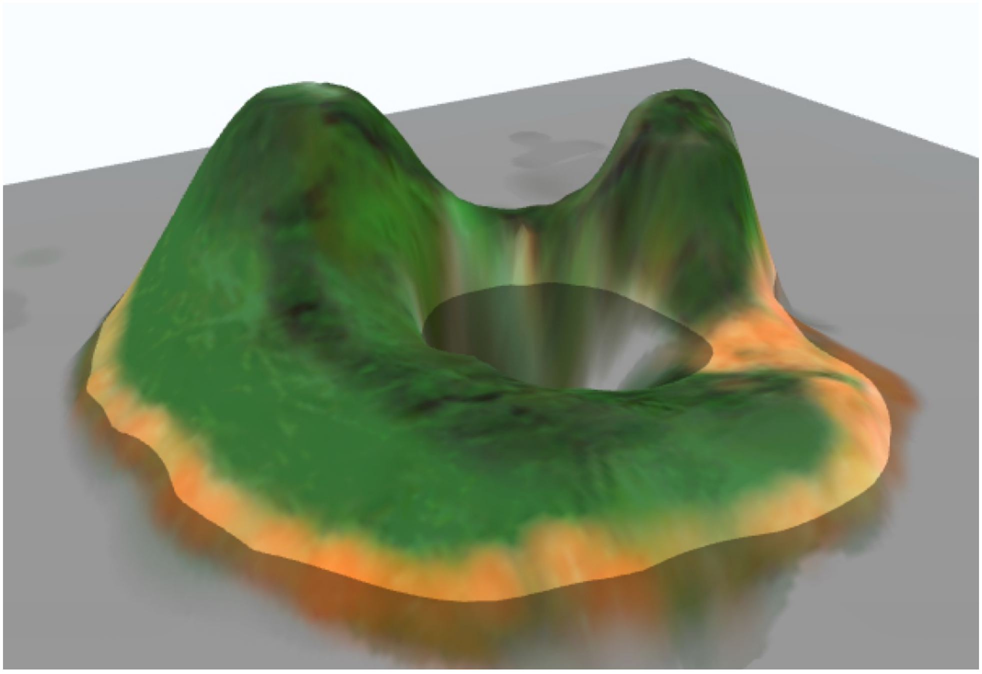
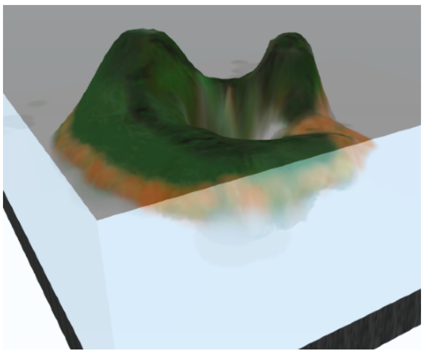


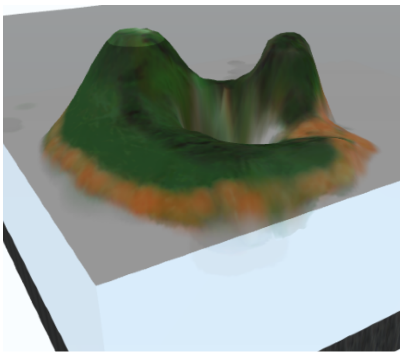
Figure 15. A landscape.

You will now research the connection between the number of islands e, the number of lakes m, the number of peaks p, the number of valleys d and the number of saddle points z. The aim of this optional item is for you to discover this connection.

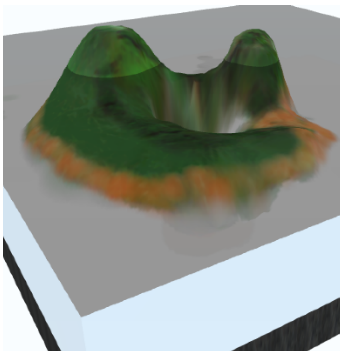
For this research you can make use of the idea of an invariant (as you already encountered in the basic assignments). Then, every time a move was made, the formula kept resulting in the same number.



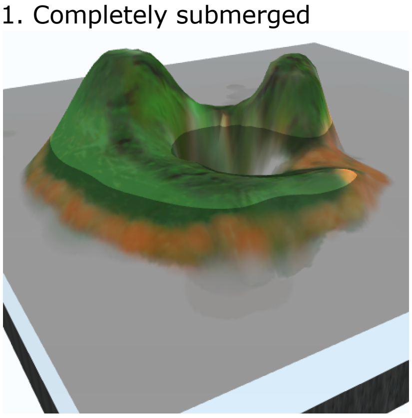
1. Completely submerged



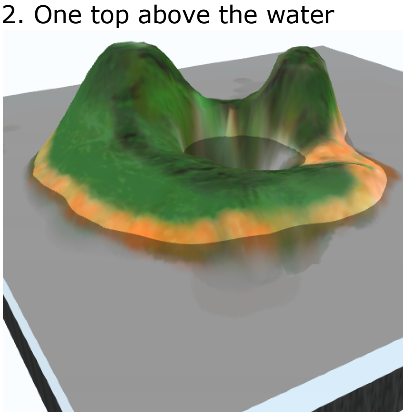
2. One top above the water



3. Two tops above the water

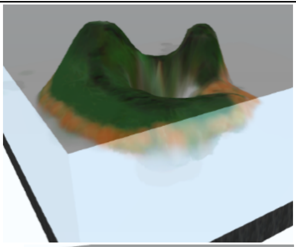
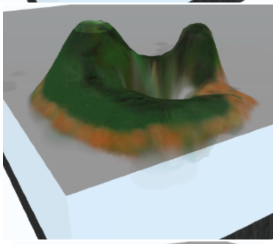
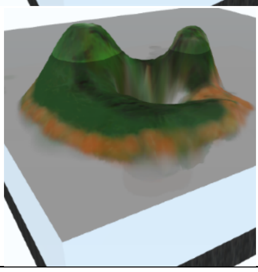


4. Connected tops



5. Completely above water level

Now we don’t do “moves” but we slowly lower the water level. For every change in water level, keep track of how the number of islands e, the number of lakes m, the number of peaks p, the number of valleys d and the number of saddle points z changes – in so far as they are above the water.

Game situation	Number of islands e	Number of lakes m	Number of peaks p	Number of valleys d	Number of saddles z
	0	0	0	0	0
	1	0	1	0	0
	…	…	…	…	…

Do this for as many landscapes as you need. Use height maps to keep it simple. You could draw a number of phenomena that are unlikely in the real world, like an exactly horizontal crater edge or plateau, or three mountain ridges that come together at exactly the same height (a so-called monkey saddle, see Figure 16), but you should avoid those situations until assignment 3.

Assignment 1. Research if there is a relation between the number of islands e, the number of lakes m, the number of peaks p, the number of valleys d and the number of saddle points z, that is the same for every (above water level) landscape.

Assignment 2. Argue that your relation is right. Hint: use invariance for every type of transition that is possible when the water level drops.

A normal saddle has room for two legs to go down; and in between, the saddles moves upwards following the horse’s back. A monkey saddle has a third depression for the monkey’s tail.

Assignment 3. Extend your relation from assignment 1, so that monkey saddles are also possible. Once you managed that, do the same for “multi”- monkey saddles: saddle points where more than three mountain ridges come together at exactly the same height – for monkeys with more than one tail.

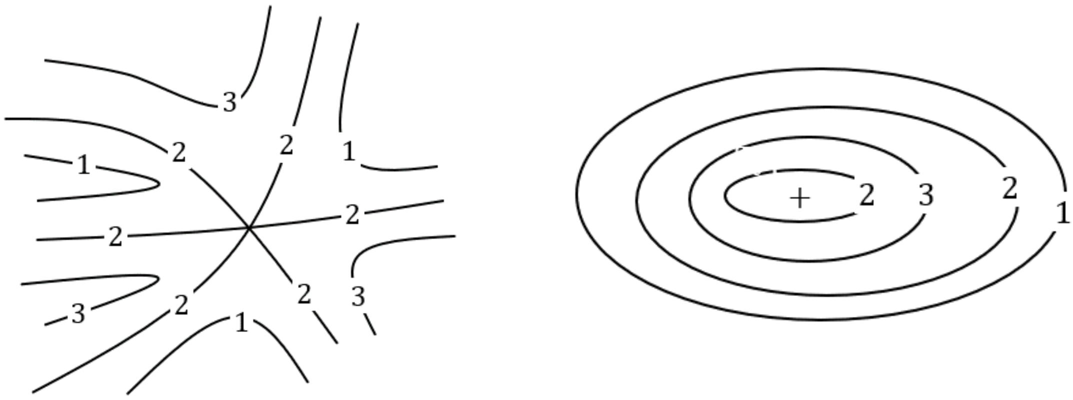


Figure 16. Height map for a monkey saddle (left) and for an exactly horizontal crater edge (right).