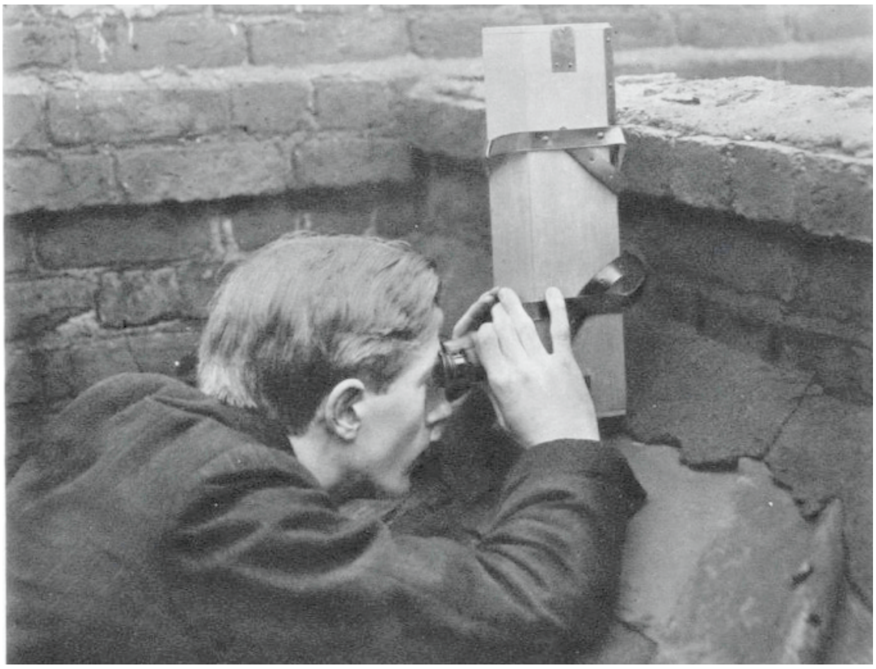
Mathematics B-day 2013

November 15, 9:00 - 16:00 hr

Reflecting on mirrors

The Periscope

This photo can be found on www.geheugenvannederland.nl



First World War: Man testing periscope to look over trench near a wall, 1915.

Exploration 1 It works like this!

Take about 10 minutes to find out how a periscope works. You can find information on internet (Google 'periscope' for instance).
Do use the words ‘mirrors’ and ‘light beams’ in your explanations and sketches about how it works; since that is what this Mathematics B-day is about.

Further introduction Mathematics B-day 2013

Today's topic

This Mathematics B-day is about mirrors and light beams. You know about mirrors. You will see at least one every day, even if you look more at yourself than at the mirror! The light beams you can't really see. But if you want to understand how mirrors work (and work together), you will need those light beams as drawn lines in a sketch or an exact construction. You probably already used them in Exploration 1. Today's theme is investigating and explaining all kinds of phenomena around mirrors.

Parts of the day

This Mathematics B-day assignment consists of three parts: *basics, own investigation, completing the final assignment*.

– The *basics* form the necessary preparation for part 2: *own investigation*. You will explore all kinds of properties of mirrors and lightbeams in single and multiple mirrors. The exploratory questions (called *Explorations*) are meant to introduce something new, but there are also questions (called *Assignments)* for which you will need to include an answer, including your mathematically correct reasoning, in the final report.
– In your *own exploration* you choose two items from research questions A, B and C. These research questions are continuations and deeper explorations of the basics. Research question D asks for a record. That one can be added too if you have time for it.
– In the *final assignment* you will describe the work you have done in your *own investigation*. Tell your story in a clear and convincing way. Of course you will use relevant illustrations. Be sure that your description can be understood by people who do not take part in the Mathematics B-day, but who do understand the necessary mathematics. That means that you will have to introduce the problems clearly, and that where relevant, you will have to refer back to what you explored and explained in the *basics*.
  In short: you will write your own clear story, supported by mathematical arguments. The style of your presentation will certainly play a part in its assessment!

Experimenting: available tools

Experimenting with real mirrors is a part of this day.

- Your teacher will hand out mirror tiles and tape to make hinged connections between them. Use the mirrors where relevant.
- There is also an applet to simulate a mirror room and other constructions with multiple mirrors. Further on in the assignment it says where you can find the applet.

Indication for your time management of the day:

- Take your time to investigate the basics of the first part to ensure that you fully understand the principles of mirroring and the techniques that come with it. Take at least 90 minutes for that.
- Make a choice which two of the three research questions A, B or C you will investigate in depth.
- The final report must be delivered digitally. If you hand in paperwork, make sure that a scan of that work is readable in the digital version.

Enjoy, and good luck with this Mathematics B-day assignment!

Basics

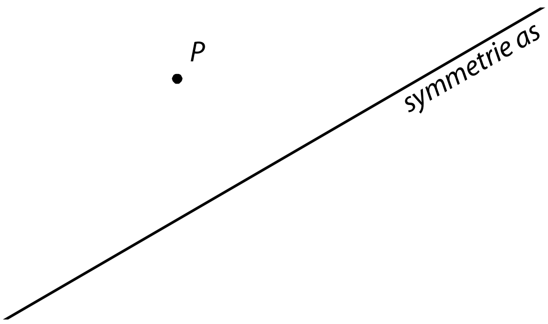
A. Relation between line symmetry and mirroring

Line symmetry

You can see (almost) perfect line symmetry in this beautiful atlas moth.
Perfect line symmetry has an axis along which you can fold a figure in such a way that the corresponding points to the left and the right of the axis will match exactly. The moth will in fact show its symmetry by simply closing its wings!

Exploration 2 Investigating line symmetry

In the figure below we take a slightly more abstract approach using a pair of compasses and a ruler, or a set square. The axis here is called the symmetry axis.
Imagine that point *P* is a point on the moth's left wing.



a. Using your set square, determine the point *P′* on the moth's right wing that corresponds to point *P*.
b. What can you say about the relation between the *symmetry axis*, the *line PP′* and the *distance* of *P* and *P′* to the symmetry axis?

P′ is called the mirror image of *P*, even if there is no mirror. Why it is called that? That will be explained next.

The mirror principle

If a light beam strikes a mirror, the mirror reflects the beam. But generally speaking the beam will not return along the same path.

The *mirror principle* will apply, though:

Angle of exit is angle of entrance.

The angles are measured with respect to the *normal*. That is the line that sits at a 90° angle to the mirror. If you look at the mirror from above, it will look the same as in figure 1.
The mirror principle is a *physical* property. We will now use this property for our reflections. From now on it will be our <u>*main reflection property*</u>.

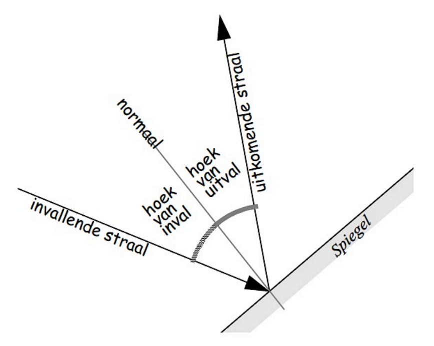


figure 1

From reflection to mirror image
In figure 2 you see a light source *L*, for example an ordinary lamp. It broadcasts light beams in all directions. Two of these light beams (***a*** and ***b***) have been drawn.
You also see a mirror. The (reflecting) front side is facing the light source.
The drawing is a top view of the situation (as in figure 1). To put it differently: we concentrate on the directions in the plane in which we draw and forget about the space surrounding us.

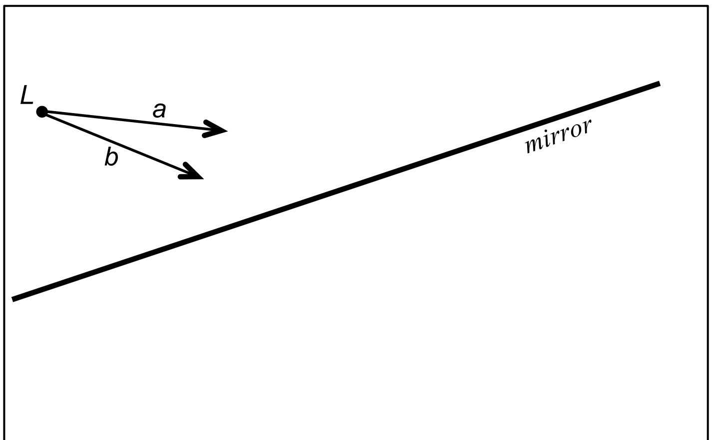


figure 2

Assignment 1 Where do the reflecting beams come from?

a. First draw the two beams ***a*** and ***b*** as far as the mirror, and apply the mirror principle. Now draw the reflecting beams for ***a*** and ***b*** in the mirror.
b. The reflecting beams are parts of lines. Draw the *whole* lines, i.e. including the parts behind the mirror and mark the intersection.
c. That marked intersection is in a special position in relation to the light source!
What position is that? Prove the correctness of your reasoning.

Conclusion
Upon reflecting a source in a flat mirror it seems as if the reflected light beams are coming from a virtual source behind the mirror. That virtual source is the image of the original source.

Everything is a light source
What you found for the lamp in Assignment 1, is also true for not-lamps. Every object you see is a light source: your eye sees an object because of the light coming from it. That light is originally coming from another source (a lamp, the sun), but that doesn't matter.

Assignment 2 Where does that beam come from?

a. In figure 3 draw the light beam coming from van object *A* via the mirror to eye *B*. The drawing must match the mirror principle!

Perhaps you drew the mirror image *A′* of *A* in the mirror in question **a.** and connected that with *B*. And you used that to find the light path from *A* to the right place in the mirror.

b. Does it also work if you use mirror image *B′* of *B*? Do you get the same result?
Support your answer with mathematical arguments.

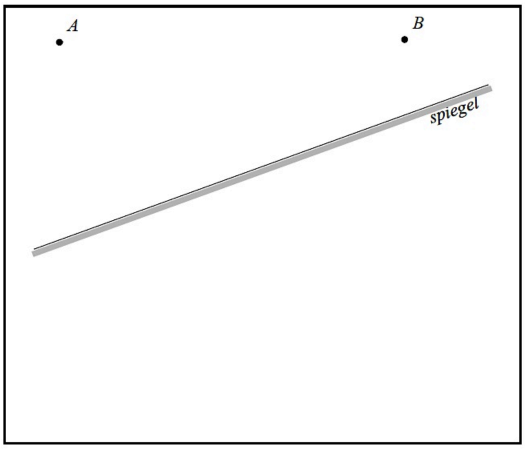


figure 3

B. More mirrors, many more images

The following assignments all involve more than one mirror. You will be dealing with reflections of reflection. The periscope in exploration 1 was already an example of that. For this part it will be useful to experiment with mirror tiles. First two examples to set the tone, followed by some theory.

Exploration 3 An initial experiment

Make a set-up as sketched in figure 4, consisting of two mirror tiles (*A* and *B*) and one mouse with a flower (or another object with clearly different left and right sides). Take care that the two mirrors are, as much as possible, perpendicular to the table and parallel to each other. The situation as shown is a front view of the set-up. The mouse is in between the two mirrors and looking at your.

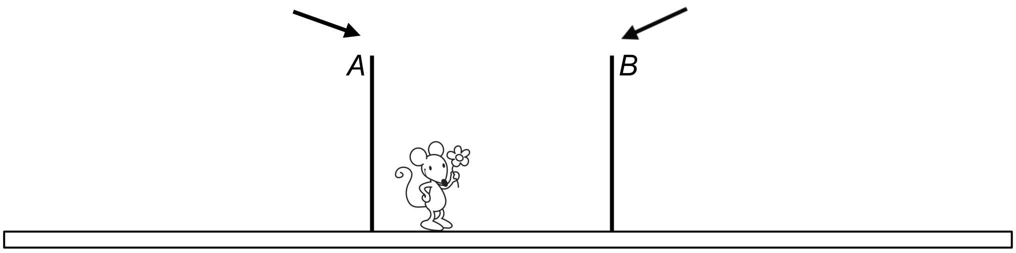


figure 4

a. If you look over the top of one of the mirrors along the direction of one of the arrows, you will see the mouse in the other mirror. Draw the two reflections of the mouse; take care that the mirror mice have the flower in the correct hand. Use **Work sheet 1**.

b. Also add reflection *B′* from mirror *B* in mirror *A*.

Mirrors *A* and *B* and the space between them, including the mouse, are part of the real world. We call that part a *basic cell*. The space between *A* and *B′* (including the reflected mouse), is the reflection (in mirror *A*) of this basic cell. It is a virtual space. We call it a *virtual cell*.

c. Add more virtual cells (reflections of mirrors and their repetition) with mice in the figure on Work sheet 1.

d. What structure can you find in that series of cells?

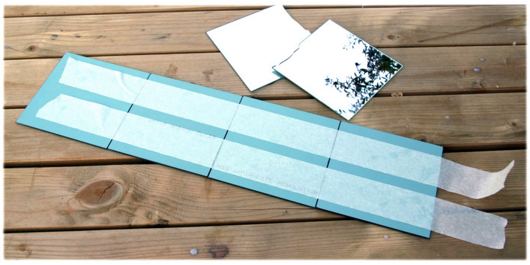
You have now found that you can generate an infinite amount of reflections with two mirrors, but that isn't all!

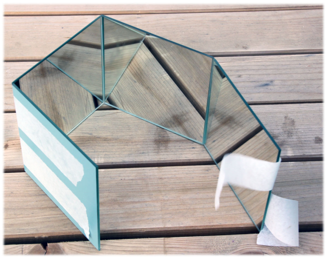
From nano-grand piano to infinity times infinity

This grand piano is made with nano-lego. It's about 5 centimeters large.

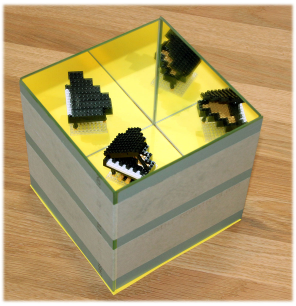
We now put four mirrors in a square. Look at the photos and read the instructions, but follow along as well. It's much better than doing it on paper, and helps you to discover things.

Put four tiles together upside down, use two pieces of tape, and carefully bend the room into a square and tape the whole closed. You now have a square mirror room.

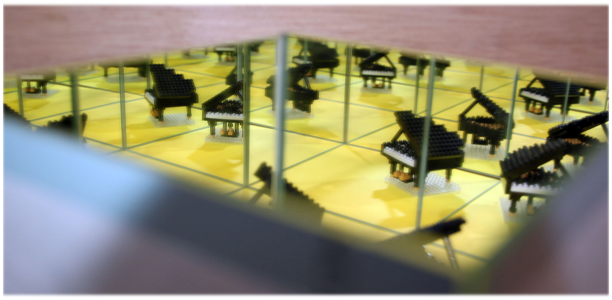




Place the square around the nano-grand piano (or use your own asymmetrical object). Of course you will see multiple reflections.

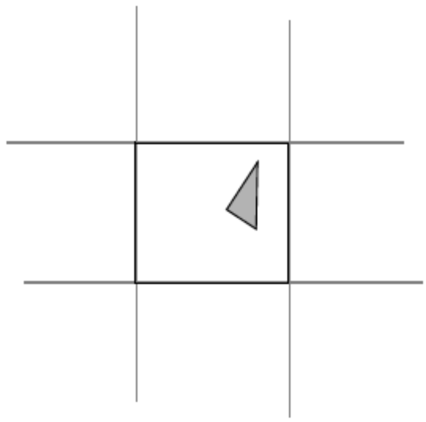


This is what you see if you look along the edge of the mirror tiles:



If you bend down a bit deeper, you will look further into the endless virtual space. Look around a bit, and conclude: the nano-grand piano's are lined up in rows with the same structure as the mouse-with-flower-images. There is an endless number of rows next to each other. A row of rows, all with the same structure.

Just as with the example with two parallel mirrors, you can take the whole here as a *basic cell* (the square formed by the four real mirrors and the original object within) and various copies (reflections and reflections of reflections) around it, filling a whole plane. Here we have drawn the basic cell, replacing the nano-grand piano with a non-symmetrical triangle. You are looking down on the basic cell and the space containing virtual cells around it from the top.



Assignment 3: Mapping the structure

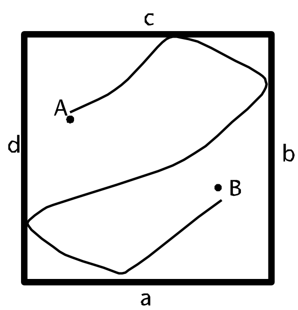
a. **Draw** the triangle in the correct orientation in the virtual cells that are given in **Work sheet 2**.

b. Which virtual cells look the same as the basic cell? Can you see regularity? Give a mathematical explanation.

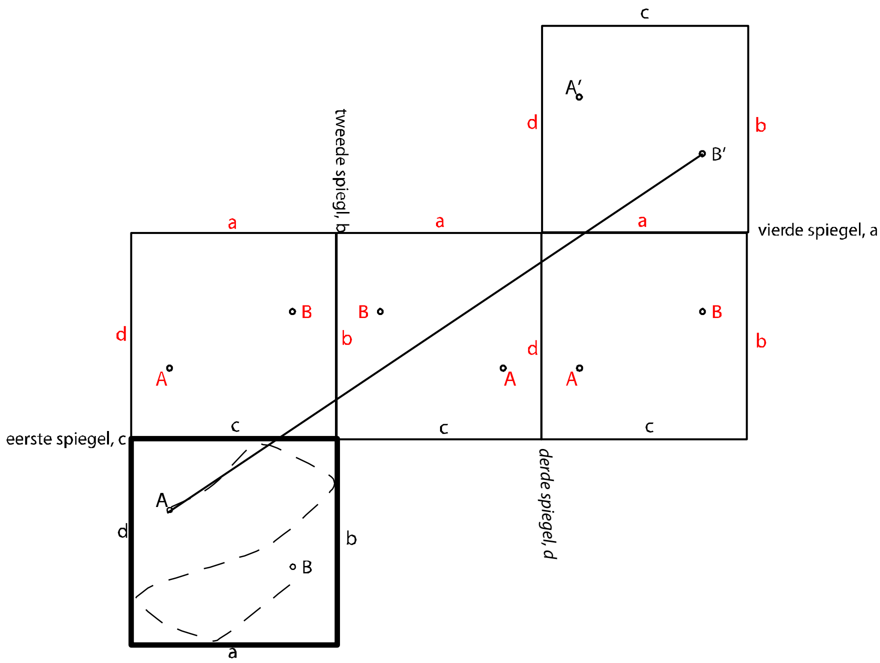
C. The FOLDOUT-method: Light paths via multiple mirrors

We continue with what we started in Assignment 2: finding light paths with given starting and ending points via one or more mirrors.
What you do with this kind of problem is to, as it were, shine a laser pen (a flashlight with a very narrow beam) via the mirrors at a point B that is in fact reached through the reflections.

Example
Here, you see a square mirror room (with reflecting walls a, b, c, d) in which you want to reach point B, starting from point A using the mirrors *c, b, d* and *a*. A sketch is easily made:



To find a precise solution, we reflect the basic cell with points *A* and *B* consecutively in sides c, b, d and a.



After the reflections have been performed, the connecting line from A to the definitive reflection B′ has been drawn.

Exploration 4 Construct the path in the original square

Complete the whole by exactly drawing the light path in the basic cell. What you do is in fact to reflect the parts of the line back to the basic cell, but now it's simple, because you can measure where they hit the mirrors!

Assignment 4 From A to A' is also possible

a. Also draw the line from A to A'. You will notice that you have to reflect via c, b, a, d. Draw the resulting light path in the basic cell. Use Worksheet 3 for that drawing.

b. If you yourself were standing in a real square mirror room, light path AA' from question **a.** indicates that you can see yourself in the mirror. But do you see yourself from the front or from the back? Or maybe even from the side?

c. Can you also see yourself after three reflections? How will you see yourself now?

Summary Foldout method and other examples

The example on the previous page goes faster if you draw a large number of squares straightaway, as if you're drawing squared paper, after which you place the reflections of the points in the virtual cells (the way it happened of its own accord in Assignment 4). Especially for squares that's an easy way.

Generally – for differently shaped basic cells, which is what we will look at now – it makes sense to start mirror by mirror. In the example it looked as if you were unfolding a folded figure, which is why it's called the FOLDOUT METHOD.

Assignment 5: Between two lines

The space between two lines (two mirrors that are connected to each other and are at an angle) can also be a basic cell for the foldout method. See figure 5.
Use **Work sheet 4** for this assignment.

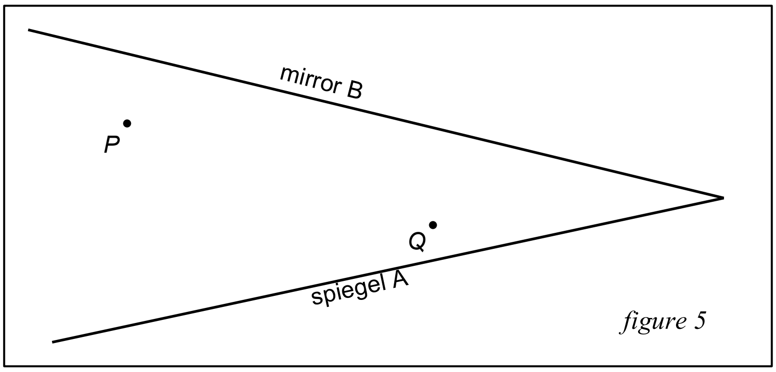


figure 5

a. Precisely draw the light path from *P* to *Q* via mirror *B* and mirror *A*. Explain your method.

b. Also draw the light path from P to P via mirrors B, A, B and A successively. Again, explain your reasoning.

In billiards the ball bounces off the cushion and then continues according to the principle *angle of entrance is angle of exit*. The centre of the ball is a point, and that point as it were strikes a line that runs just in front of the actual cushion. Seen like that, billiards is a form of reflection, and so the principle of the foldout method can be applied here too.

Figure 6 shows a billiard table from above. You must strike ball *A* in such a way that it only hits ball *B* after having first hit three cushion. You may hit one cushion twice, as long as cushions are hit three times.

figure 6

Exploration 5 Three-cushion billiards

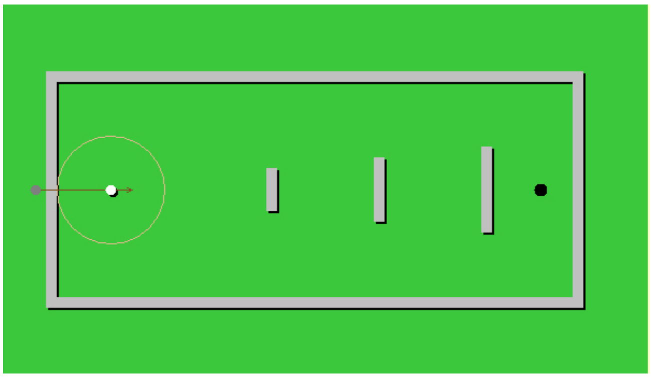
Construct several different routes for ball *A*, in which ball *B* is only hit after ball *A* has hit three cushions.

You can also look at minigolf in this way.

Exploration 6 Minigolf (on rekenweb)

a. Play the game online on the website to become familiar with it http://www.fisme.science.uu.nl/toepassingen/03015/opgave4.html.

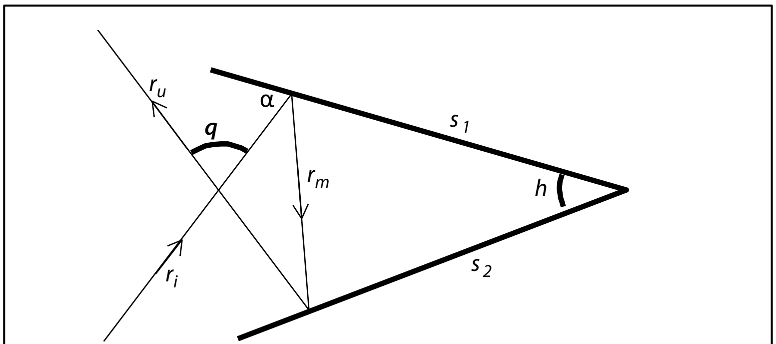
b. Use *precise* drawing to find the correct direction to hit in this figure.
Are there more solutions?



D. Light paths for multiple mirrors

Assignment 6 Bouncing in two mirrors

Imagine a light beam, as in figure 7, being bounced by two mirrors s_1 and s_2.



figuur 7

The angle between the two mirrors is $\boldsymbol{h}$; the angle between incoming beam r_i and mirror s_1 is $\boldsymbol{\alpha}$. The angle between the incoming beam r_i and the outgoing beam r_u we call $\boldsymbol{q}$.
The question is whether there is a relation between the angles h and q.

Look at this situation: $h = 40°$ and $\alpha = 60°$.

a. How large is angle q in this situation?

b. What will change if the incoming light beam has an angle of $\alpha = 80°$ with $h = 40°$?

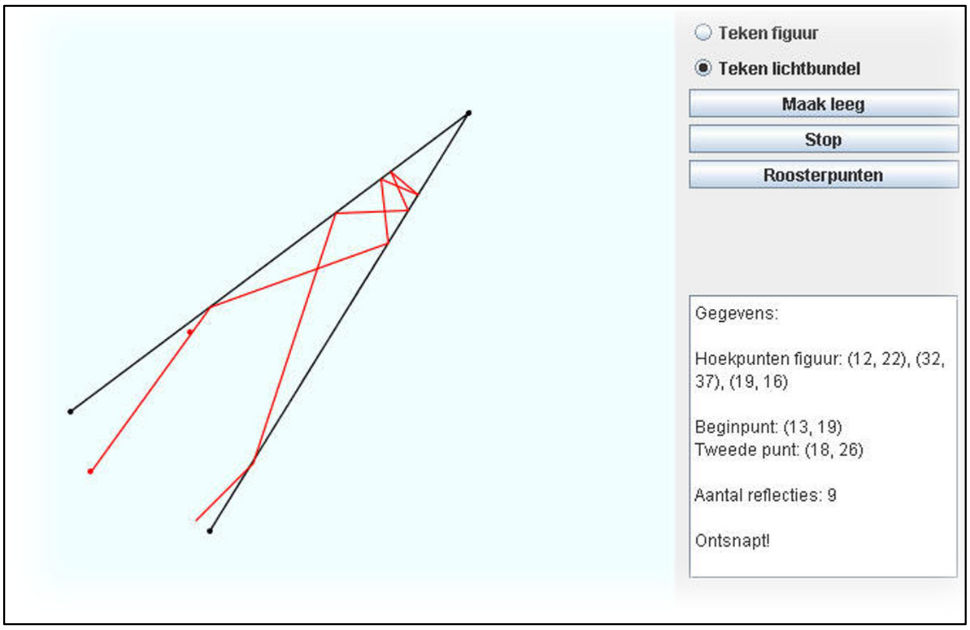
c. Look at some more numerical combinations of angles h and α.
Formulate a conjecture about a relation between angles h and q.

d. Try to demonstrate generally (so not just using numerical examples) that your conjecture is correct.

e. If the mirrors are under a certain special angle h, the light beam will come back in exactly the opposite direction. What does that mean for angle q and what does h have to be?

Experimenting digitally

On www.fisme.science.uu.nl/toepassingen/28005/test.html an applet is available with which an open or closed mirror room can be designed. 'Draw figure' can be used to design a mirror room and 'Draw beam' can be used to select two points (*Starting point* and *Second point*) which determine a light beam entering the room. The computer then draws the light path.
An example:



The applet also shows the coordinates of the points that are hit by the light beam and the coordinates os starting and second point.
The button 'Gridpoints' have to be used when you are only allowed to use integer coordinates (like in Main Assignments C and D).

What you should know further:
- 'Clear' will clear the whole screen. Your mirror room is gone then. If you want to test another light beam in the same room, activate the button 'Draw beam' and select two new points for another beam.
- If you want to adjust an existing mirror room, activate the button 'Draw figure' and then you can adjust the room by dragging its vertices.
- The 'Stop' button is needed when a light beam is repeating itself.

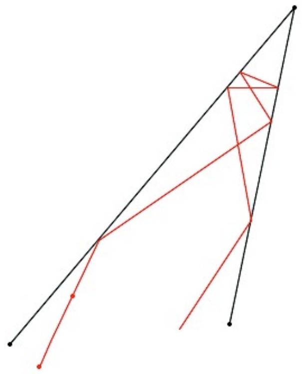
Also important:
- With 'Draw figure', the applet is able to draw an open room. If you want a closed room, make sure that you draw the last point close to the starting point. You close the room by dragging the last point onto the first point of the room.
- If in an open room a light beam hits one of the vertices, is not clear how the beam should continue from there. You will see the message 'Stuck in vertex' then.
- It may happen that the applet gets stuck. In that case, use Reload to start the applet again.

Own investigation

Select two of the main assignments A, B and C in this part.
As an extra, you can also have a go at D which asks for a record breaking attempt.

Main assignment A: Two mirrors at a (small) angle

Place two mirrors perpendicular to the table; using tape, the mirrors have been attached to each other so that they can hinge and you can therefore vary the angle between the two mirrors.
If you place an object between the mirrors, you will immediately see a number of reflections, depending on the angle between the two mirrors.



Guiding questions for your investigation:

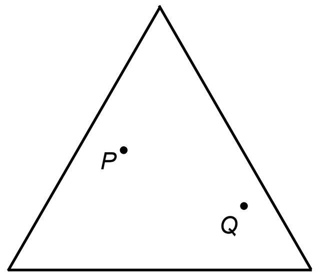
- What is the relation between the number of reflections and the angle between the two mirrors?

- If the set-up with the mirrors is large enough, you can also se yourself in the mirror(s) several times. How many times? Does that also depend on the angle between the mirrors? And if so, how?

You can use the applet to experiment for this assignment, since the applet will tell you after which number of reflections a beam will escape from the angle.

Main assignment B: Light paths in triangular mirror rooms

In Assignments 3 and 4 you investigated the structure of the infinite virtual space for a square mirror room.
In the triangular mirror room in figure 8, which has three sides that are the same size, you will investigate how light paths behave through bouncing on the walls 1, 2, 3, 4 or more times. Here, an example of the basic cell, the real space between the three mirrors, has been drawn, with two points in it: *P*, the starting point of the light path, and *Q,* the target point of the light path.
Of course, points *P* and *Q* can be freely chosen within the triangle.



figuur 8

Points to pay attention to in your investigation

- Is it possible to create a light path in which a light beam returns to the same point after bouncing 1, 2, 3 or more times, and so keeps repeating? Does that depend on the positions of *P* and *Q* inside the basic cell?
- If point Q coincides with point *P* you can investigate whether you, if you were standing in this mirror room in point *P*, would see yourself after 1, 2, 3, ... bounces in the mirror walls. If that does happen, what would you see of yourself (front, side or back)?

Further research questions might look into

- Which properties that you studied for an equilateral triangle are valid for random triangular mirror rooms?

Main assignment C: Light paths in a rectangular mirror room

This assignment involves either a rectangular billiards table or a rectangular mirror room. It doesn't really matter which. Because you can make good use of the applet for this assignment, we will stick to reflecting.
Tip: switch on the GRID in the applet. You can set up the sizes exactly.

Example:
A rectangle of 24 by 15 dm. The ball (light beam) is sent on its way from the corner at the bottom left corner of the figure, under an angle of 45° with the sides.
Finally, it turns out that the ball (light beam) ends up in one of the other three corners of the billiard table.

a. In which corner? After how many reflections along the walls?

b. A billiard table of 24 by 14 dm or one of 24 by 16 dm will give a different result.
Do take the same starting situation for your light beam: starting from the corner at the bottom left corner of the figure, under an angle of 45°.

c. Investigate the options for other sizes of the billiard table, and whether you can find a general rule. It would be great if you could simply calculate which corner the beam will end up in for given sizes of the sides.

In your investigation you may be able to make good use of the foldout method.
Using coordinates may come in usefully; set point O(0, 0) in the starting point of the ball (beam). Then the other corner points will always have multiples of the sides of the billiard table when you unfold.

Perhaps you can make good use of the 'greatest common divisor' and the 'least common multiple'. At the very least you might look up the terms online.

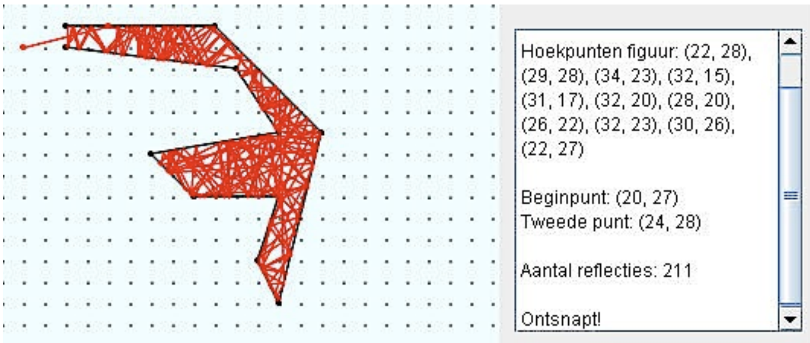
Suggestions for further investigation:

d. What happens if you don't start from a corner or on a light path coming from a corner?

e. What happens if you start under a radically different angle, but do head in the direction of another point with whole number coordinates?

Main assignment D: Record attempt number of reflections.

Here you see an example of a reflecting polygon in the applet, where a light beam enters through an entrance, and also exits after a large number of reflections.



So far it has not been possible to send in a light beam through an entrance into a polygon in such a way that the light beam could not exit again.
The light beam always came out, though after a very large number of reflections.

Your record attempts

- Design a reflecting polygon (with an entrance) with the *largest* possible number of reflections before the beam exits agian, preferably with a small number of reflecting walls. Set the applet to GRID, since you can only use whole number coordinates for the corner points and the beam in the competition.
- In your search you will keep track of the polygon's coordinates so you can experiment further with them later on; the applet will help you with this.
- If you think you have found a record, you will of course also explain why you chose this particular shape for the mirror room.

Please note! Yu have to try to get the largest number of reflections using the *smallest possible* number of mirror walls. You must also be able to explain your choices of shape and number.
Include a screenprint of the record in your report.